Aspects of the Bethe Free Energy

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Information Theory Research Group
Hewlett-Packard Laboratories Palo Alto

POA Workshop, Santa Fe, NM, USA, September 2, 2009

Motivation

Theorem (Yedidia/Freeman/Weiss, 2000)

Fixed points of the sum-product algorithm (SPA) correspond to stationary points of the Bethe variational free energy (BVFE).

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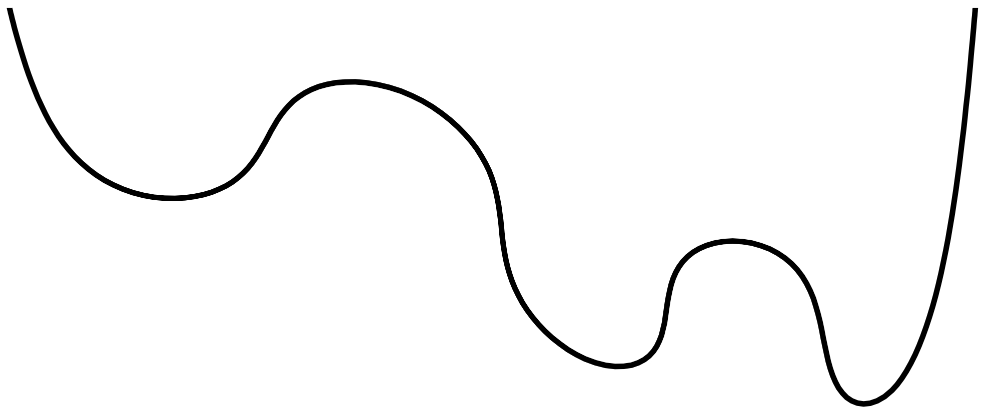

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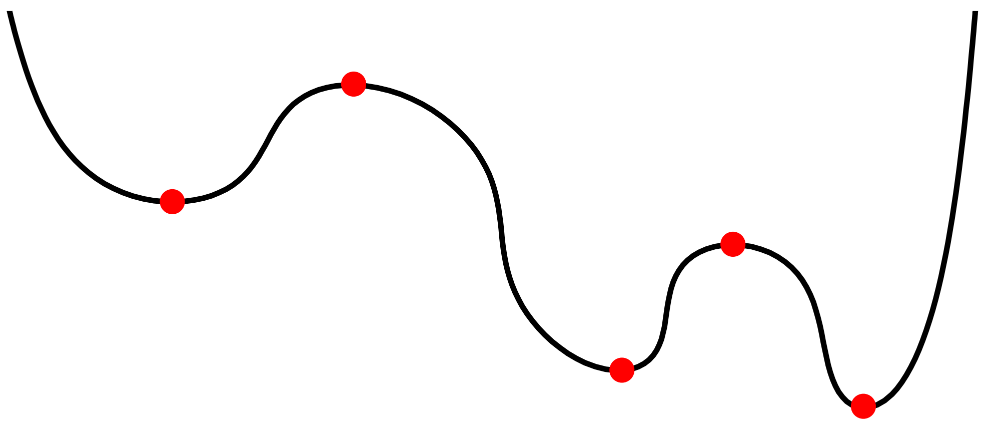

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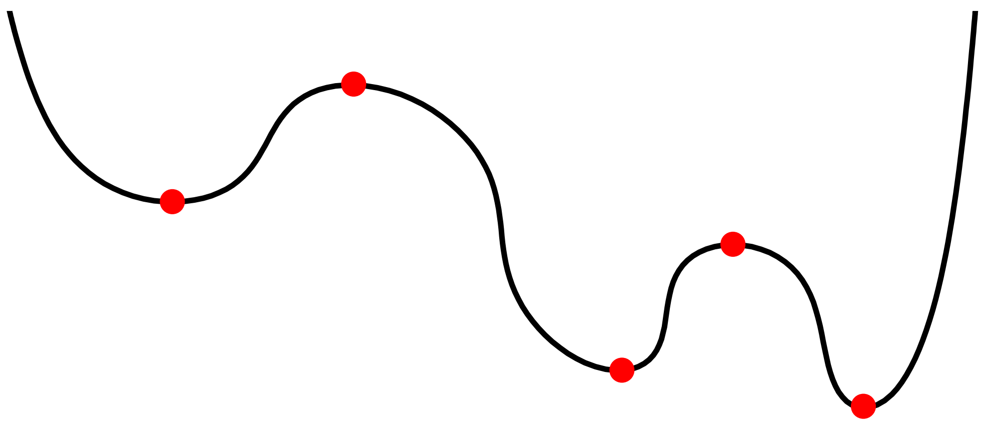

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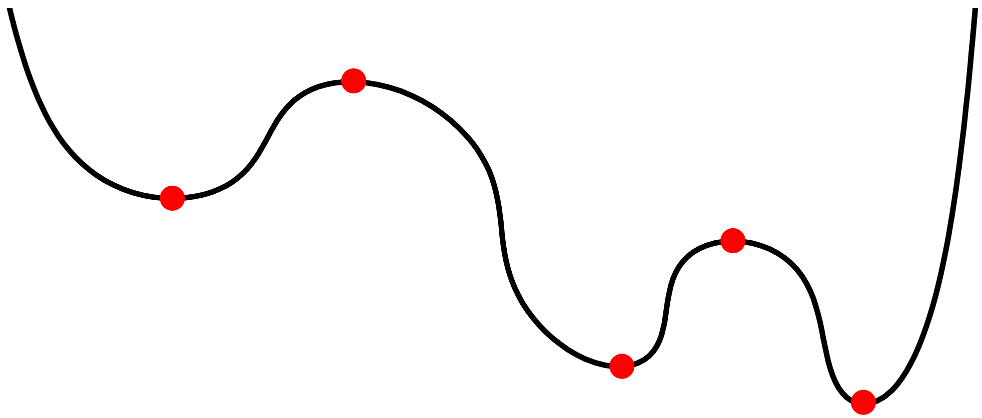

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Graphical representation of a code

Binary Linear Codes

Let $\mathbf{H}$ be a parity-check matrix, e.g.

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The code $\mathcal{C}$ described by $\mathbf{H}$ is then

$$\mathcal{C} = \left\{ (x_1, x_2, x_3, x_4, x_5) \in \mathbb{F}_2^5 \;\middle|\; \mathbf{H} \cdot \mathbf{x}^{\mathrm{T}} = \mathbf{0}^{\mathrm{T}} \pmod 2 \right\}.$$

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A vector $\mathbf{x} \in \mathbb{F}_2^5$ is a codeword if and only if

$$\mathbf{H} \cdot \mathbf{x}^{\mathrm{T}} = \mathbf{0}^{\mathrm{T}} \pmod{2}.$$

Binary Linear Codes

This means that $\mathbf{x}$ is a codeword if and only if $\mathbf{x}$ fulfills the following two equations:

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In summary,

$$\begin{aligned} \mathcal{C} &= \left\{ (x_1, x_2, x_3, x_4, x_5) \in \mathbb{F}_2^5 \;\middle|\; \mathbf{H} \cdot \mathbf{x}^{\mathrm{T}} = \mathbf{0}^{\mathrm{T}} \text{ (mod 2)} \right\} \\ &= \left\{ (x_1, x_2, x_3, x_4, x_5) \in \mathbb{F}_2^5 \;\middle|\; \begin{array}{c} x_1 + x_2 + x_3 = 0 \text{ (mod 2)} \\ x_2 + x_4 + x_5 = 0 \text{ (mod 2)} \end{array} \right\}. \end{aligned}$$

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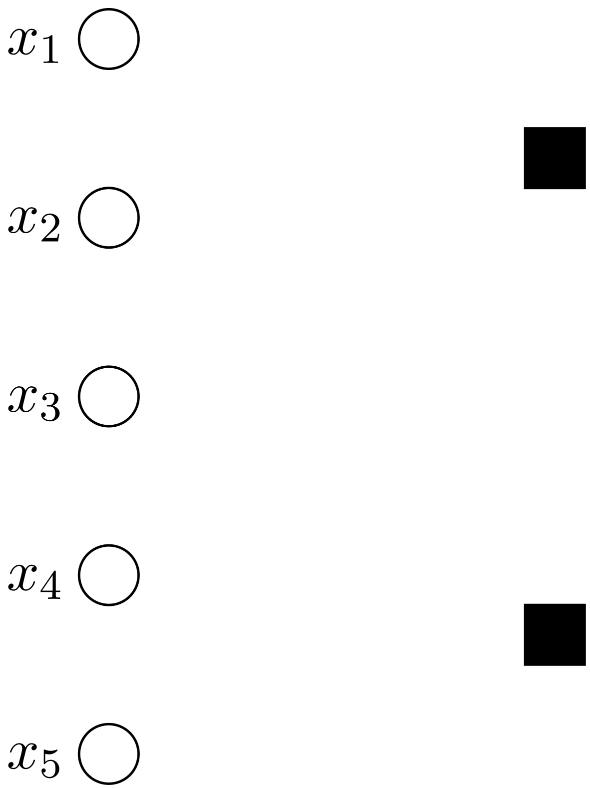

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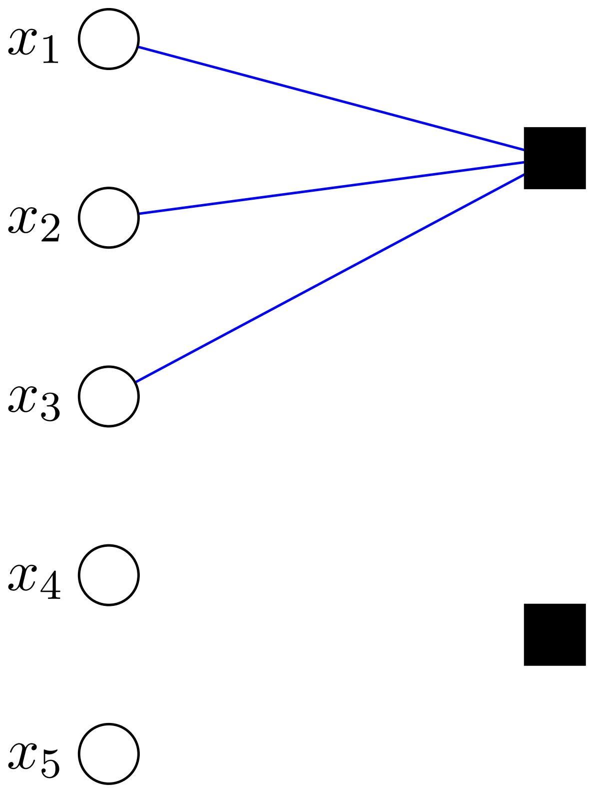

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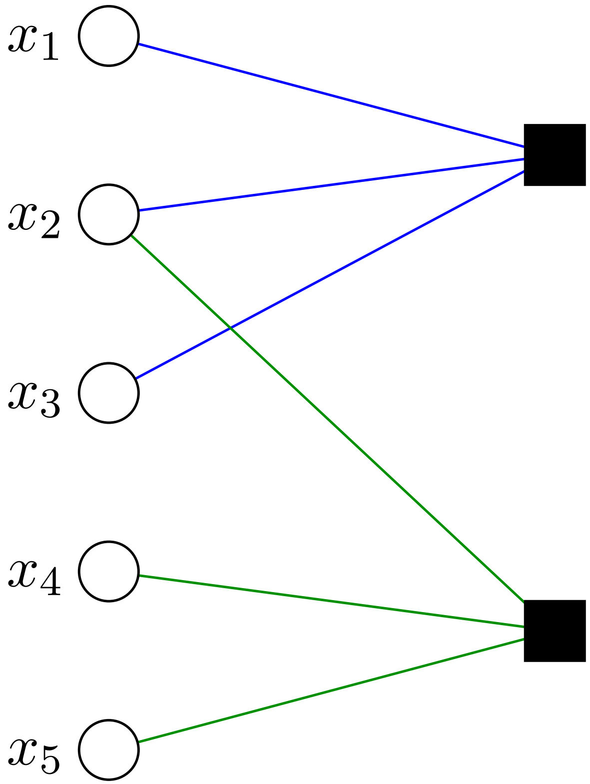

FG of a Data Communication System based on a Parity-Check Code

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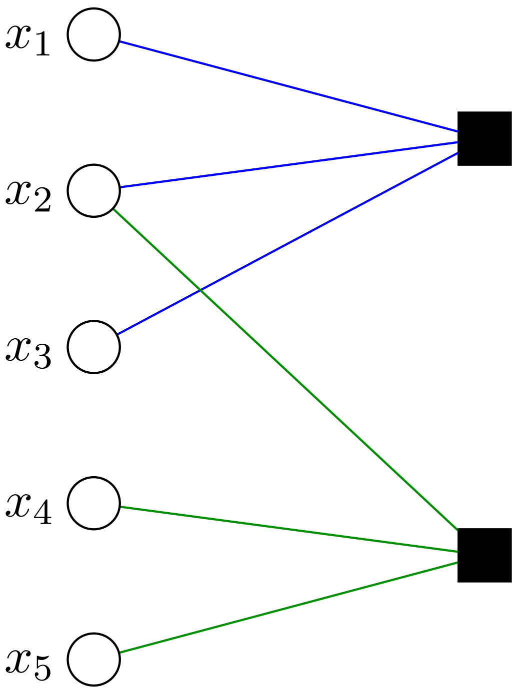

$$f_{\text{XOR}}(x_1, x_2, x_3) \;=\; [x_1 + x_2 + x_3 = 0 \pmod 2]$$

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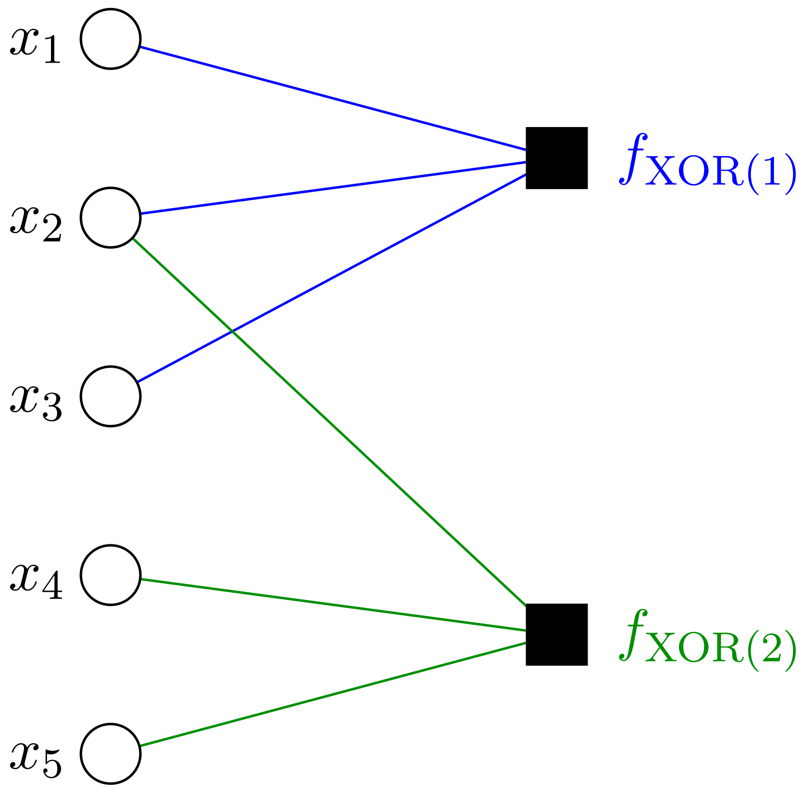

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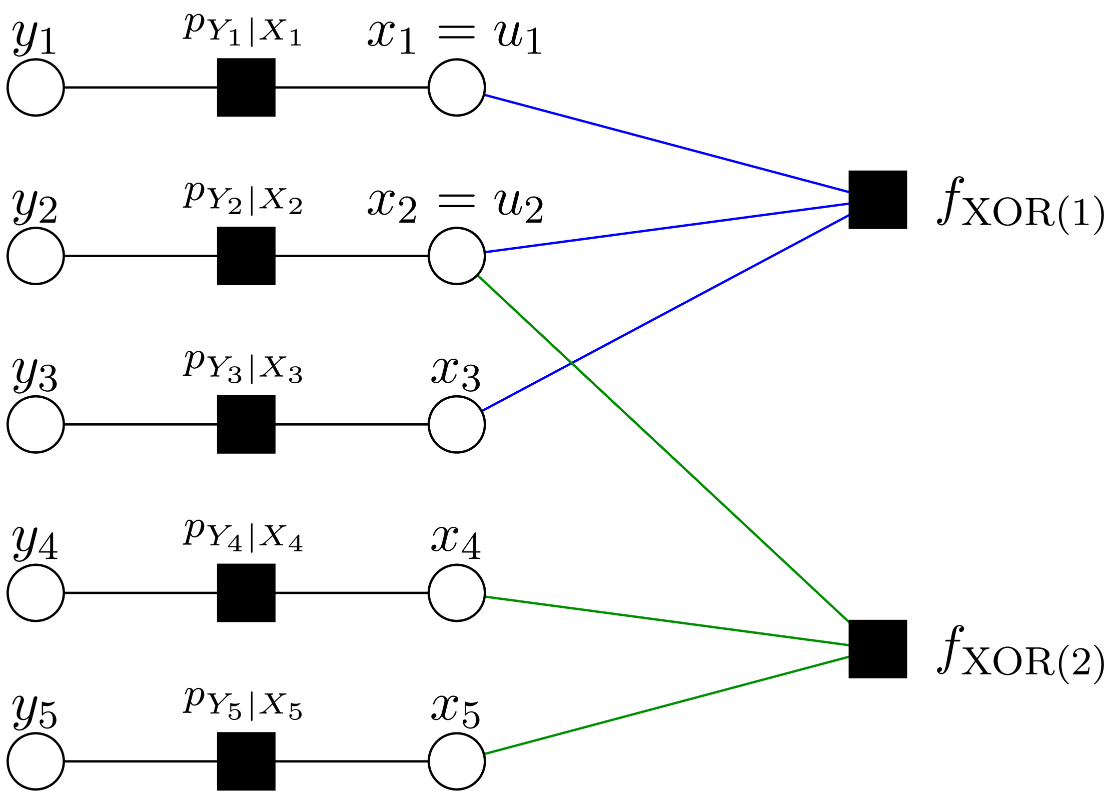

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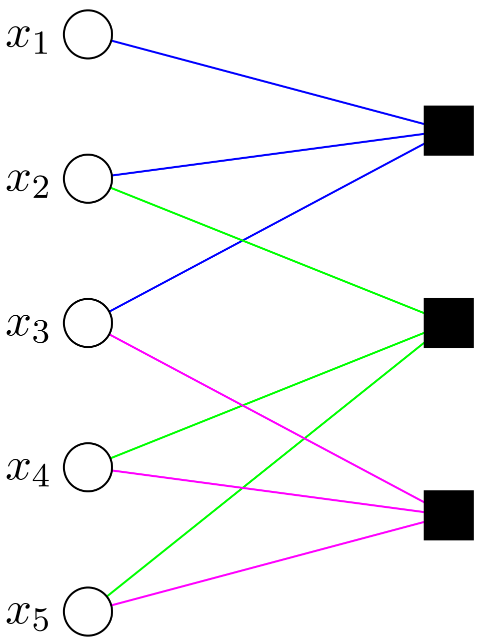

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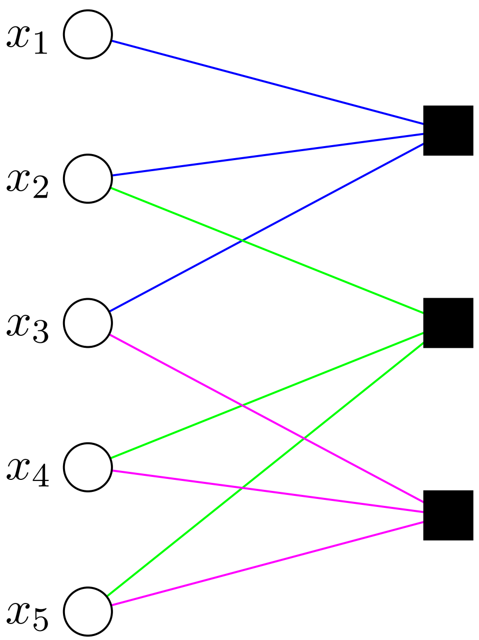

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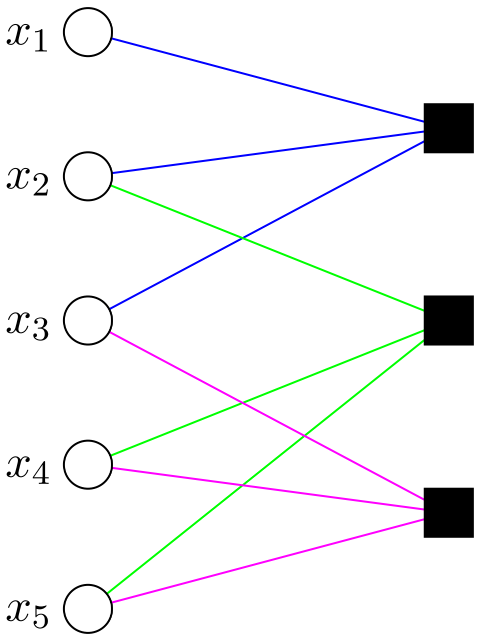

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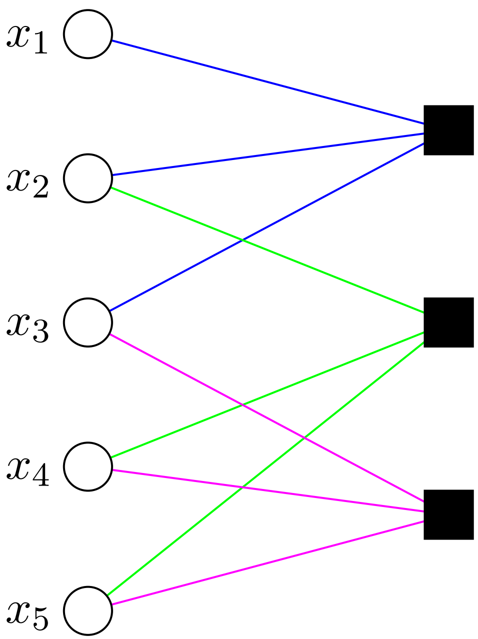

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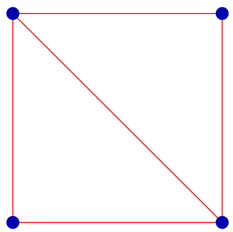
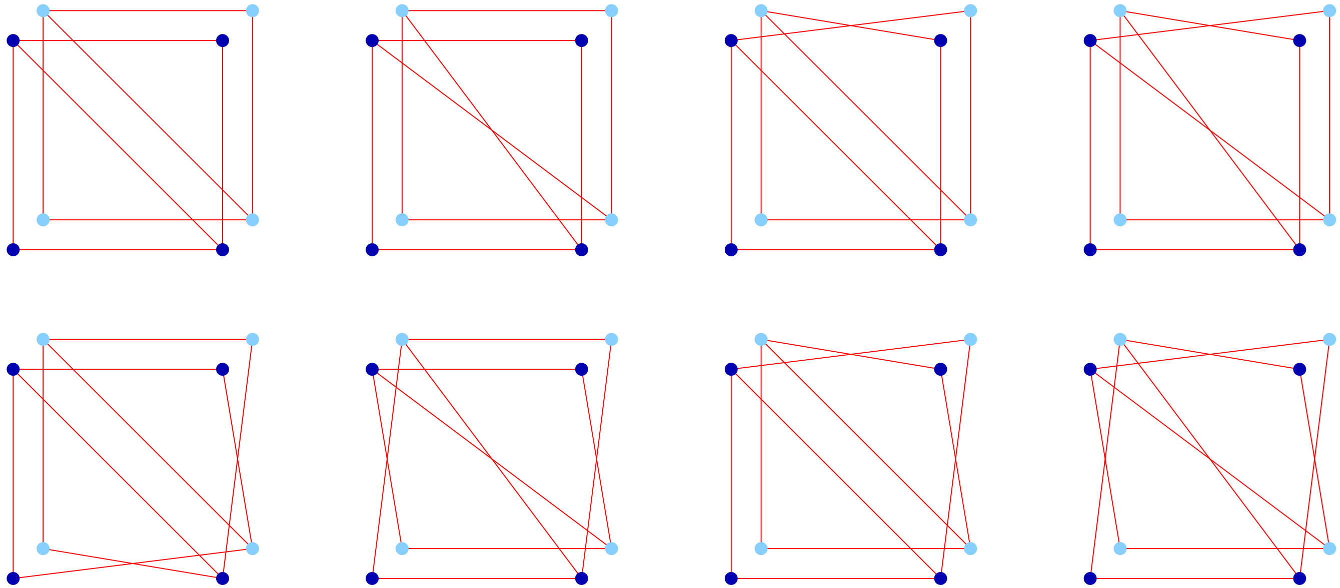
- A (j, k)-regular LDPC code is a code whose Tanner graph has uniform variable node degree j and uniform check node degree k.
- One can show that Tanner graphs of good codes have cycles. (We assume bounded alphabet size and bounded subcode complexity.)

Graph covers and their relevance for message-passing iterative decoding

Graph Covers



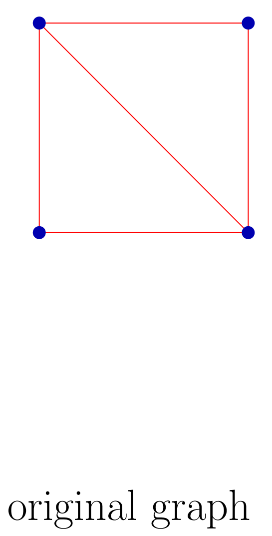
original graph

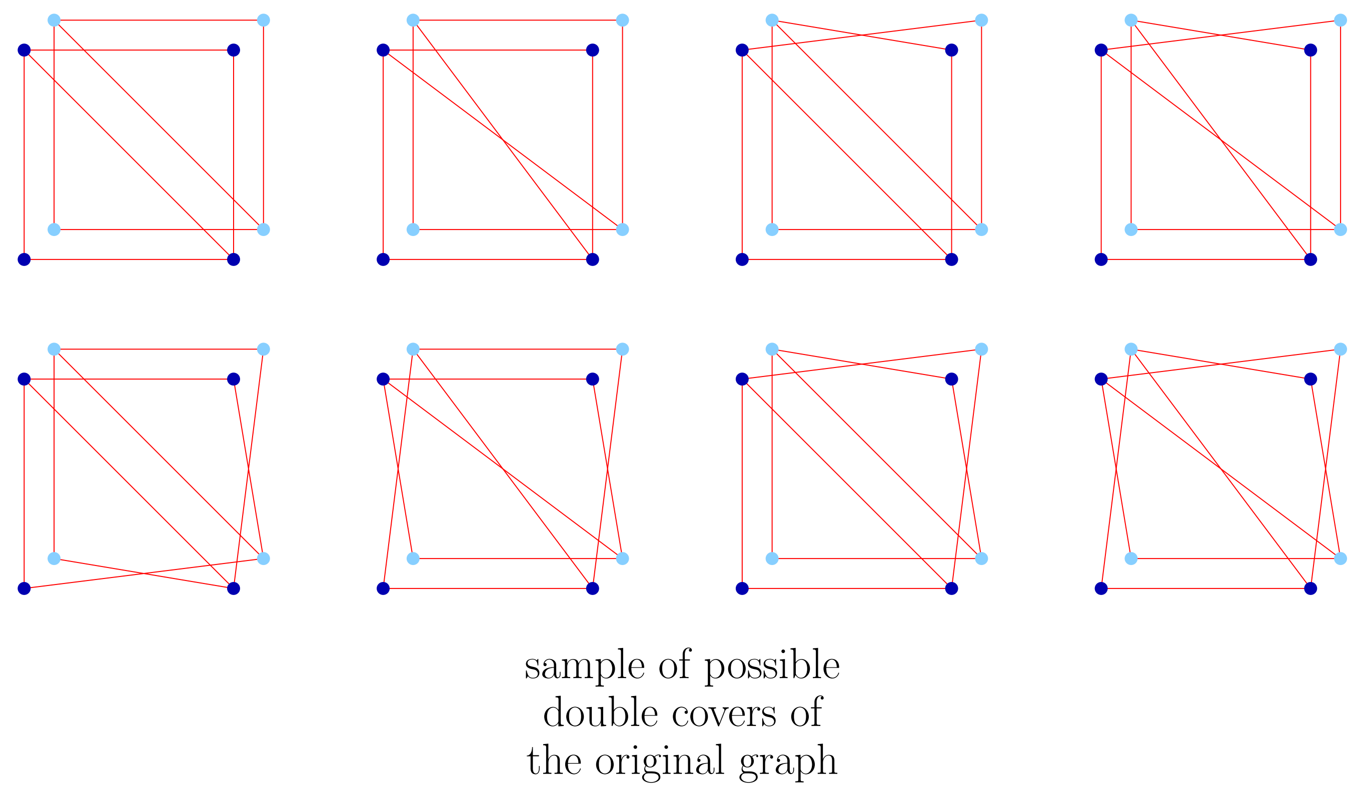


sample of possible
double covers of
the original graph

Definition: A double cover of a graph is . . .

Graph Covers

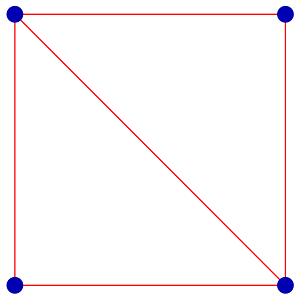




Definition: A double cover of a graph is ...

Note: the above graph has $2! \cdot 2! \cdot 2! \cdot 2! \cdot 2! = 32$ double covers.

Graph Covers



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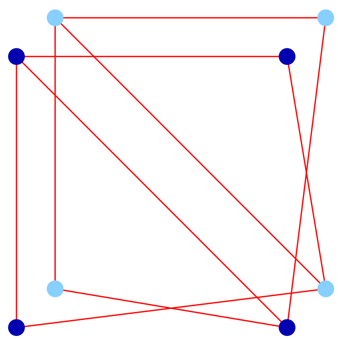

(a possible)
double cover of
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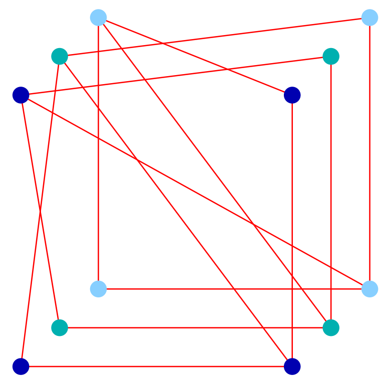

(a possible)
triple cover of
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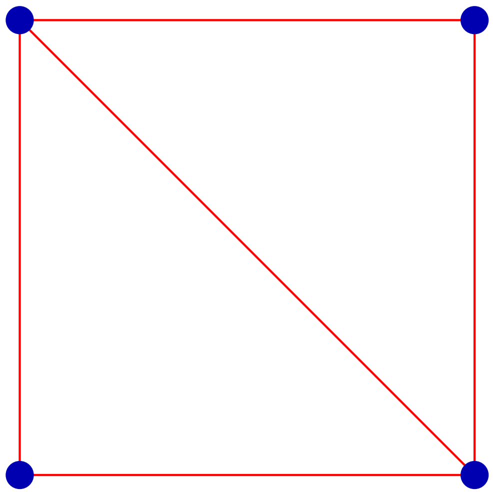
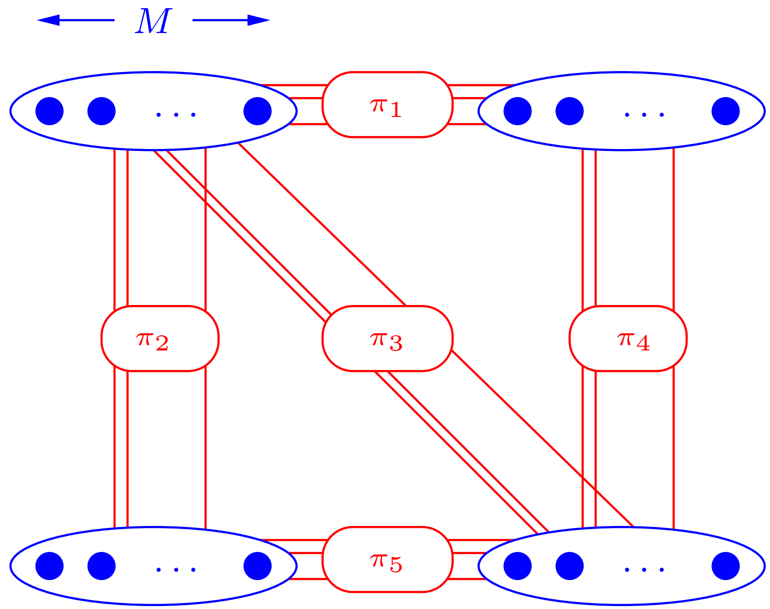
$\ldots$

Besides double covers, a graph also has many triple covers, quadruple covers, quintuple covers, etc.

Graph Covers



original graph



(possible)
M-fold cover of
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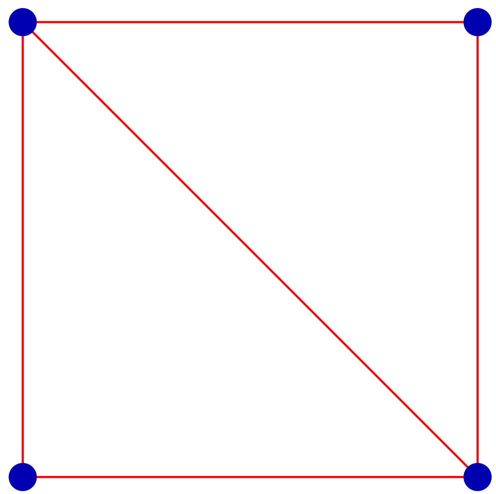
An M-fold cover is also called a cover of degree M. Do not confuse this degree with the degree of a vertex!

Graph Covers



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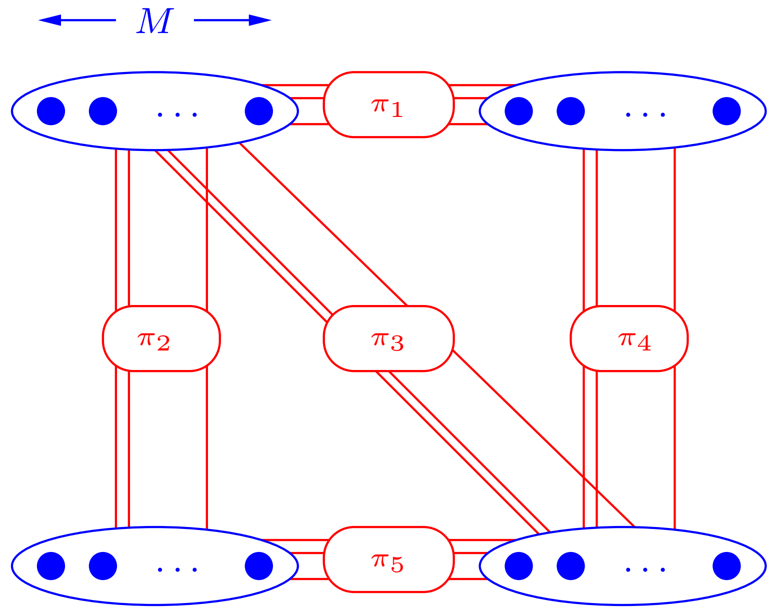

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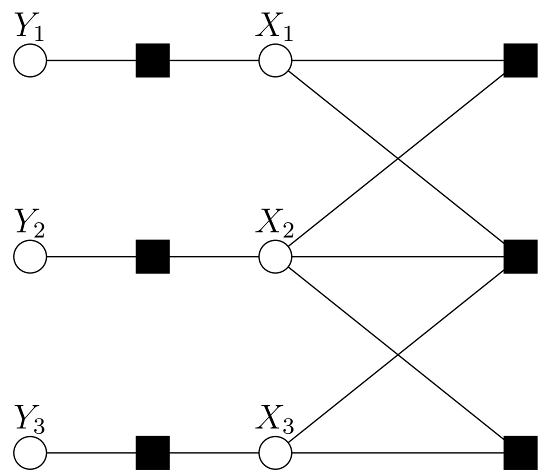
Note: a graph G with E edges has $(M!)^E$ M-fold covers.

Message-Passing Iterative Decoding

Consider this factor graph:



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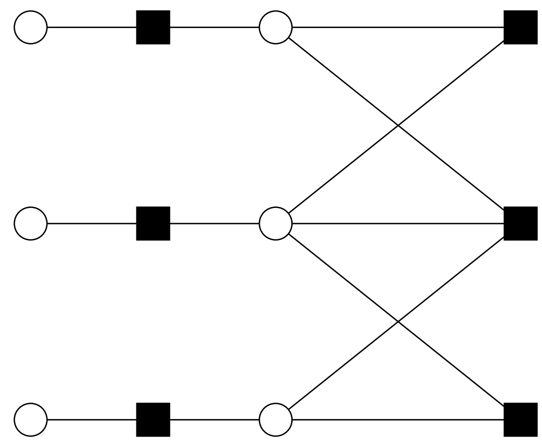

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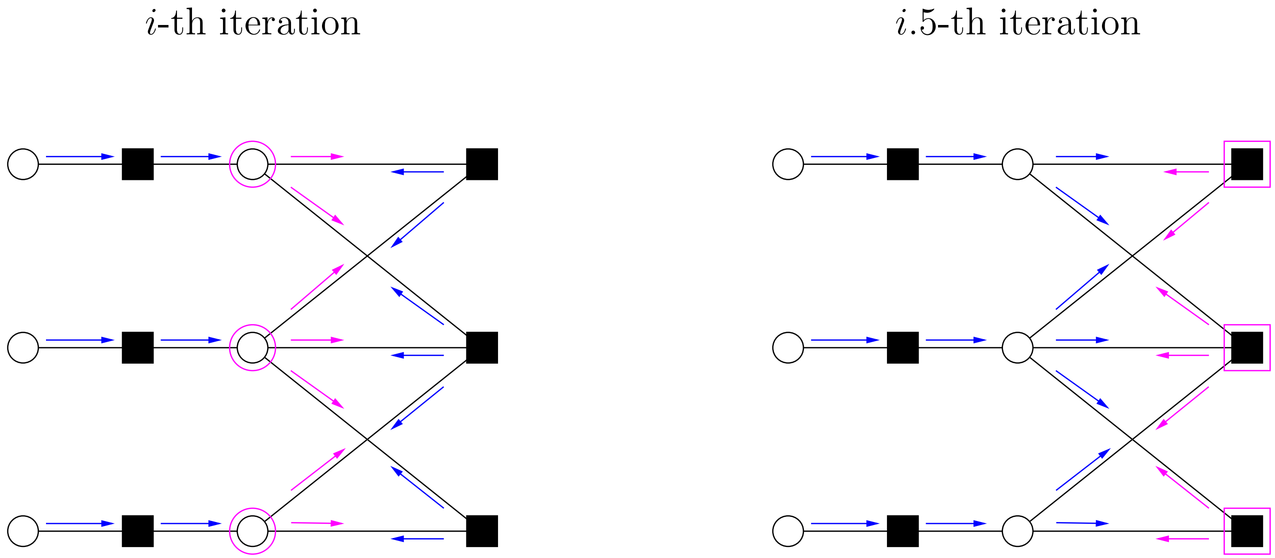

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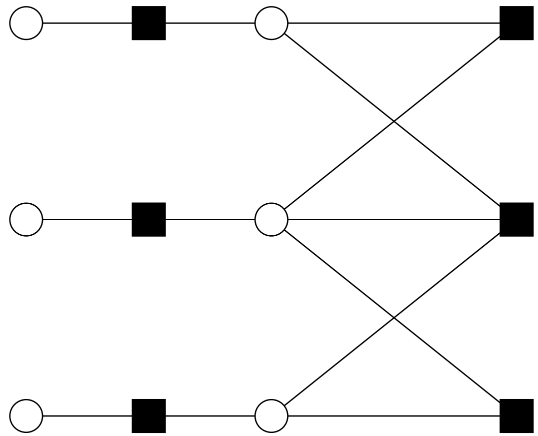

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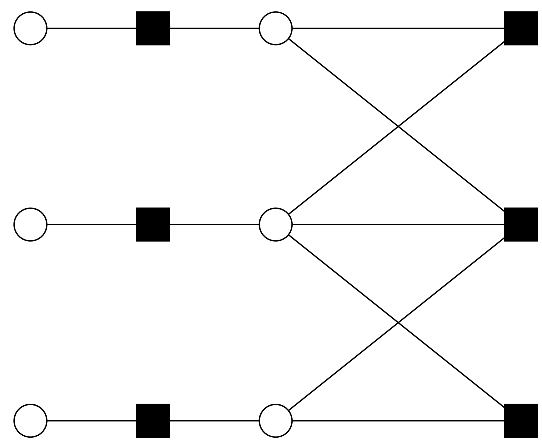
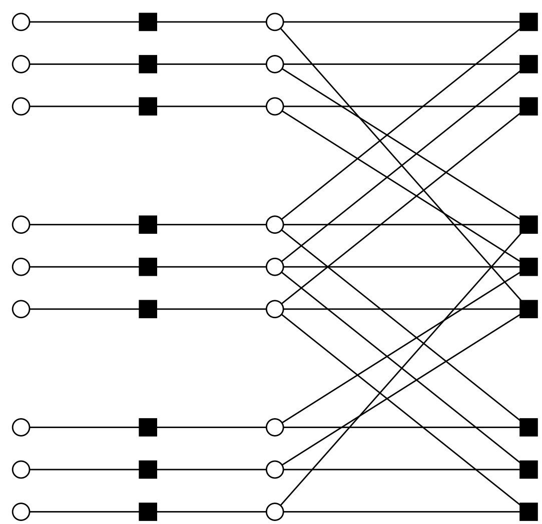


Here is a so-called triple cover of the above factor graph:



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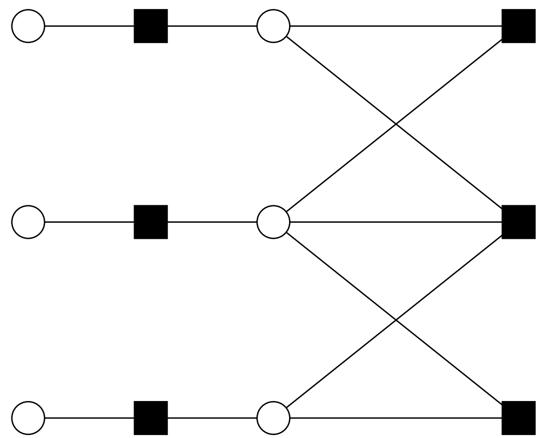

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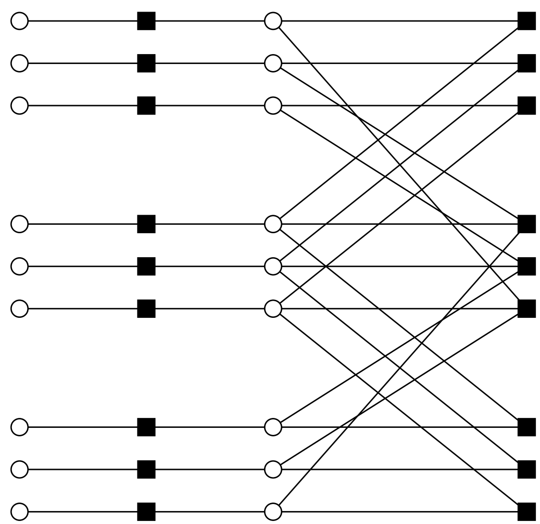
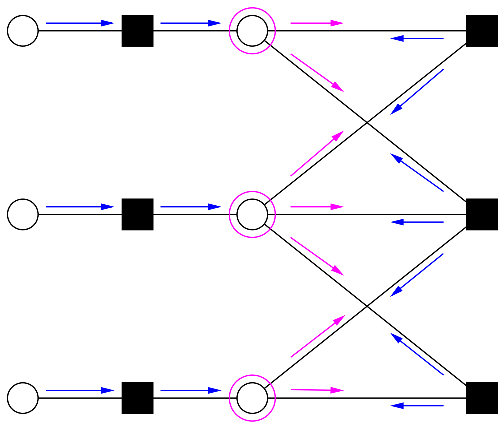


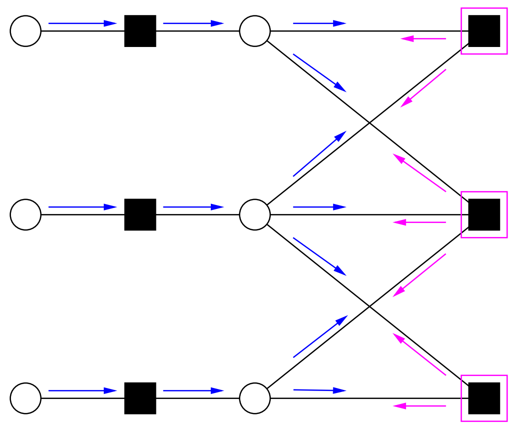
Why do factor graph covers matter for MPI decoding?

Message-Passing Iterative Decoding

i-th iteration



$i.5$-th iteration



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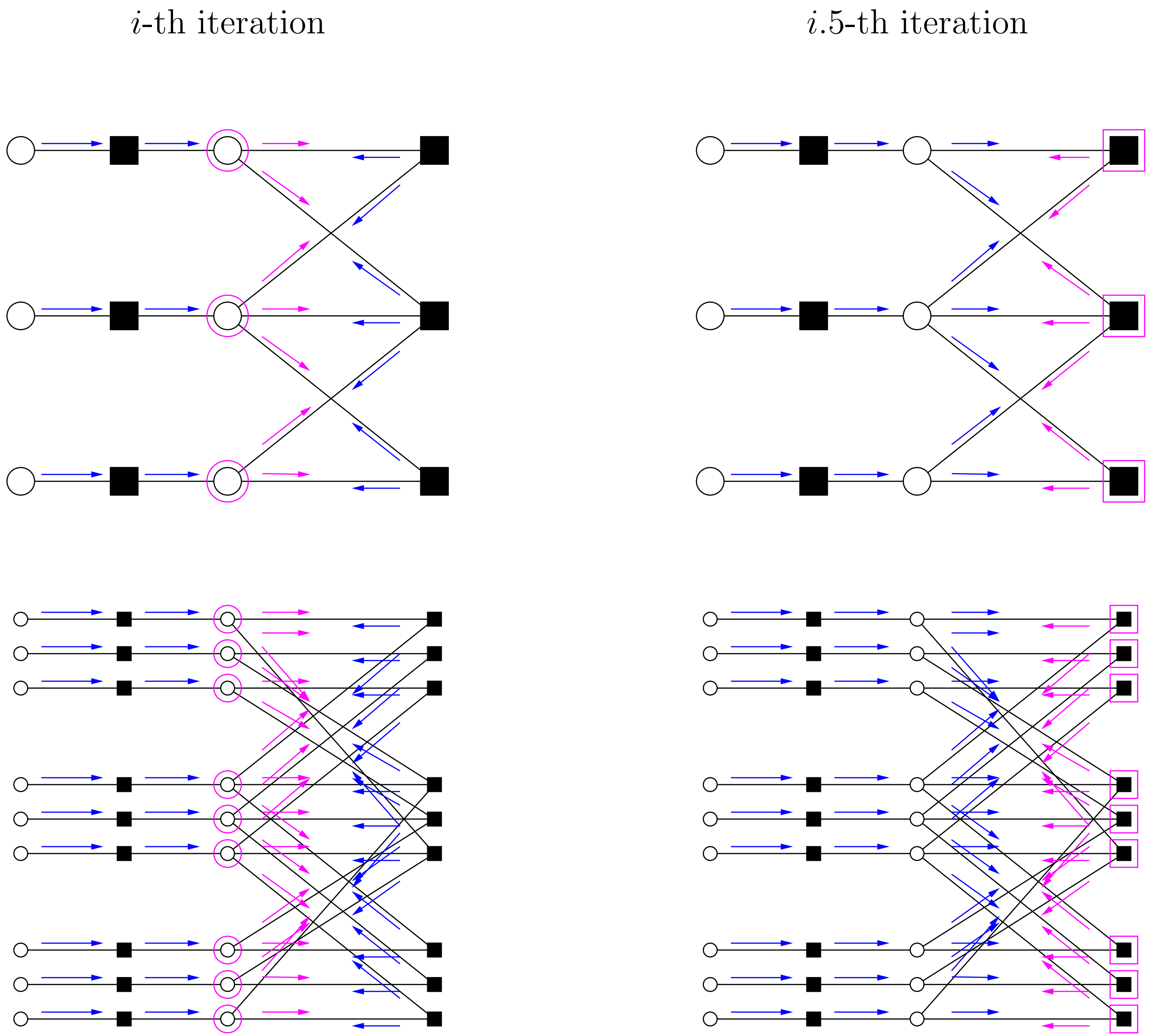

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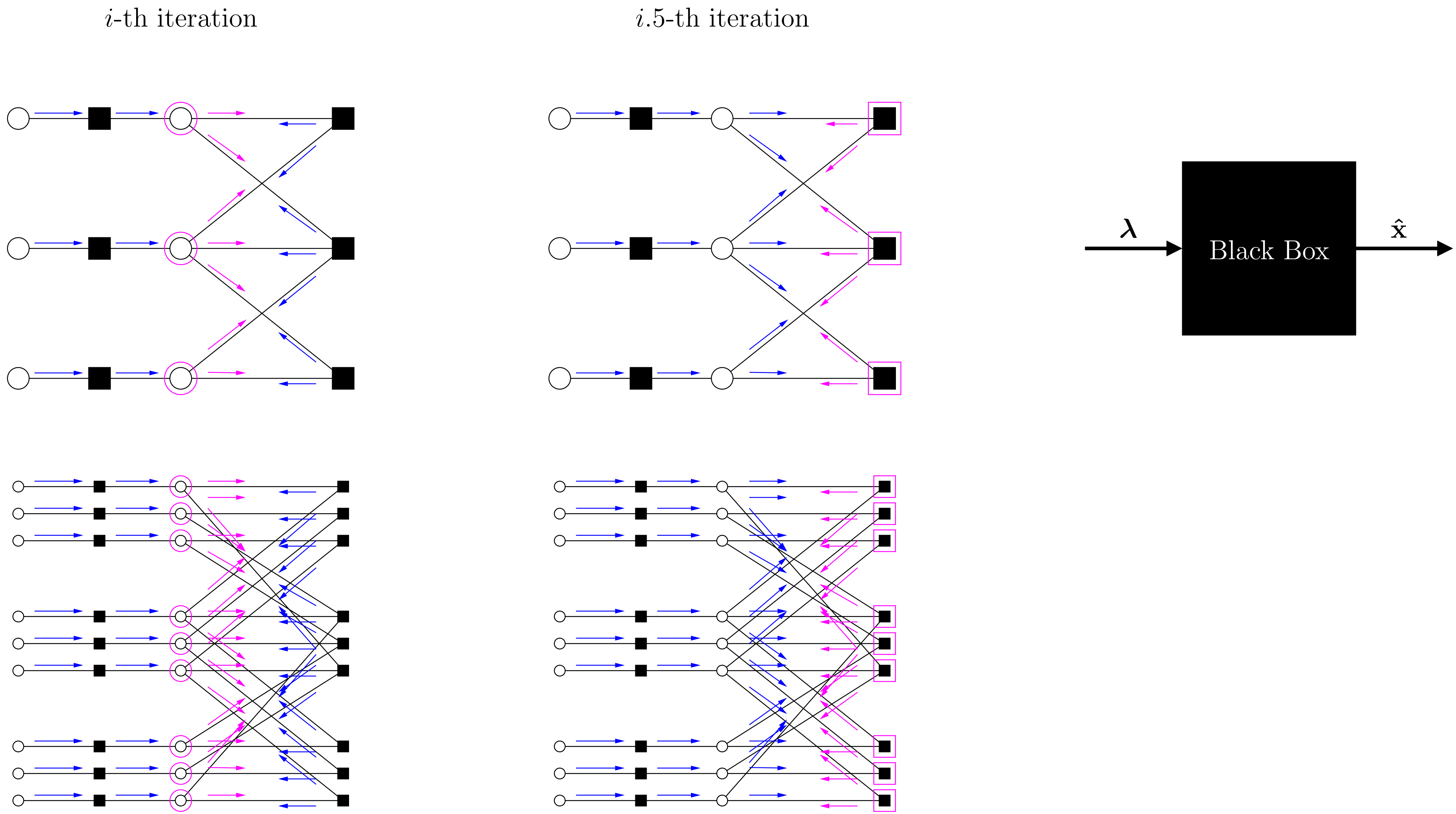

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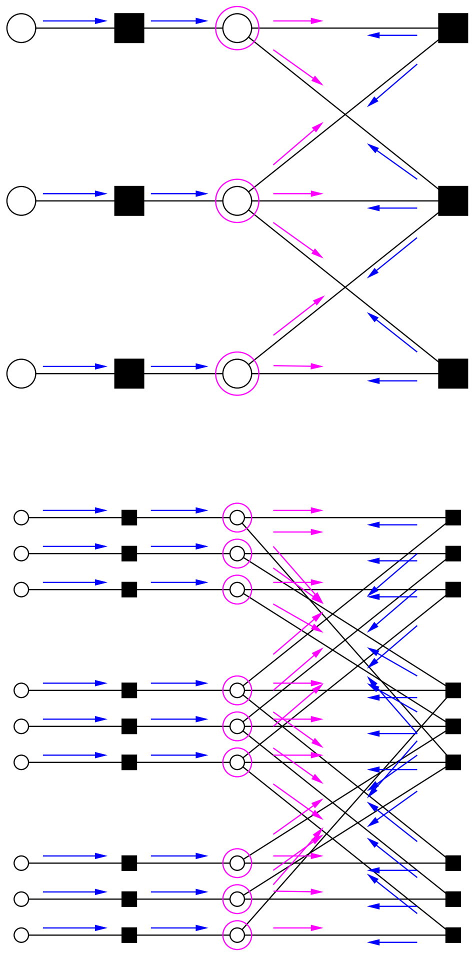

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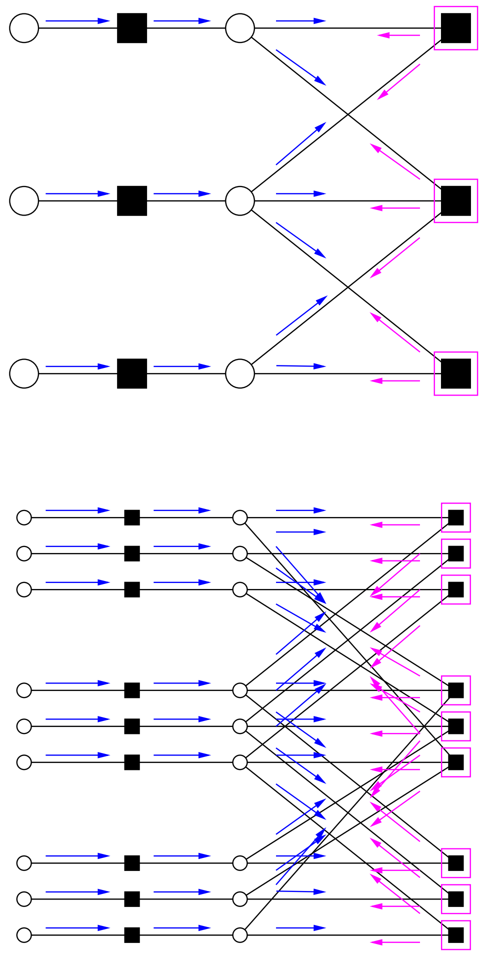
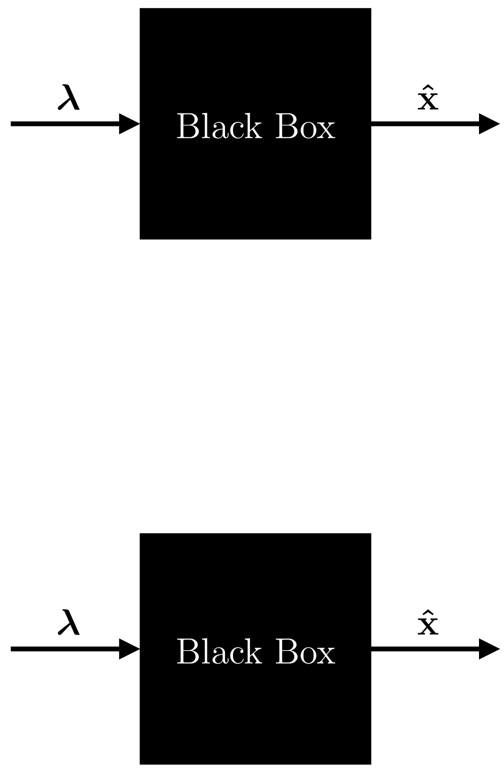




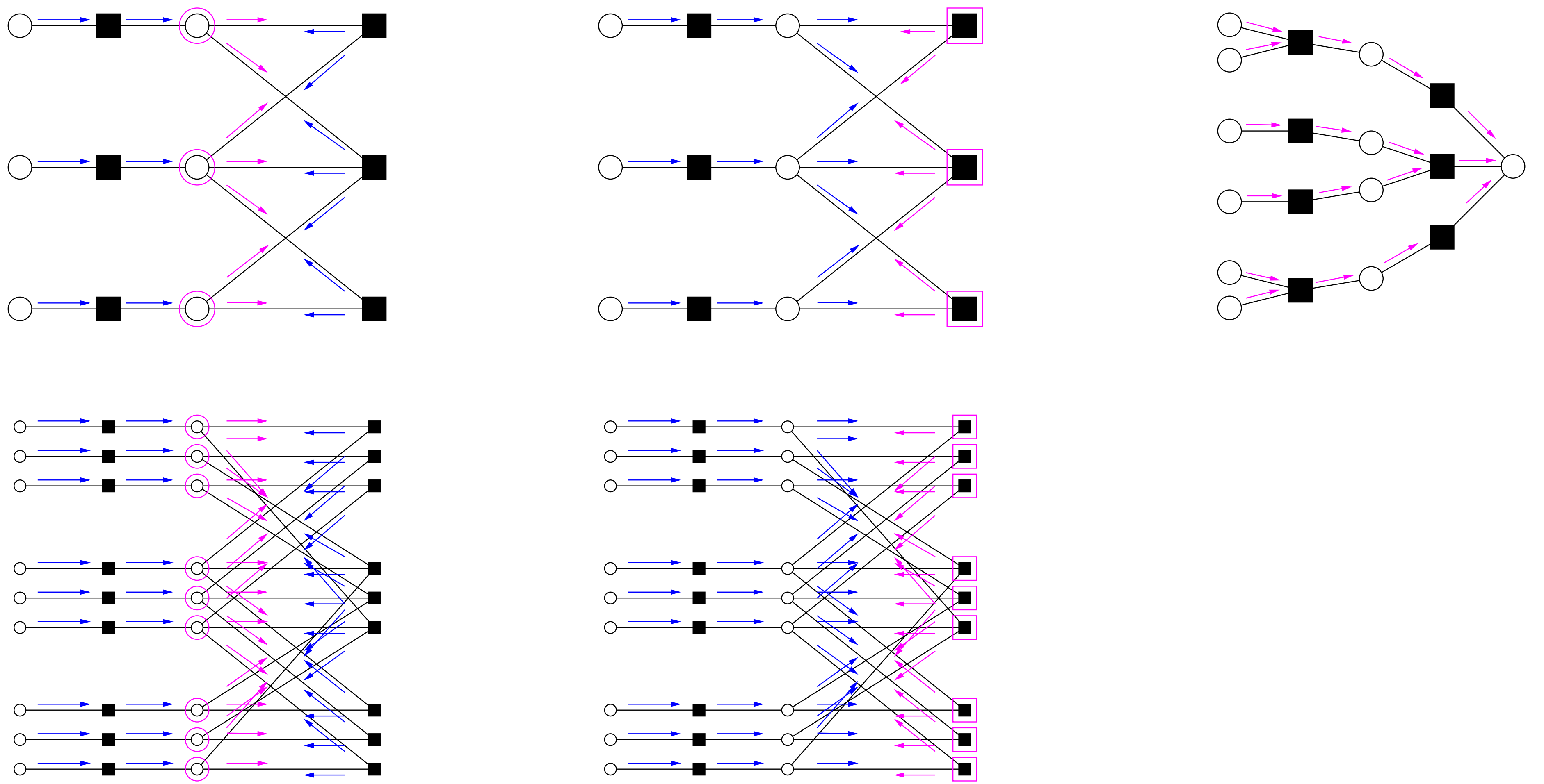
Message-Passing Iterative Decoding

i-th iteration

$i.5$-th iteration

computation tree (without channel function node

. . . where root is bit node 2



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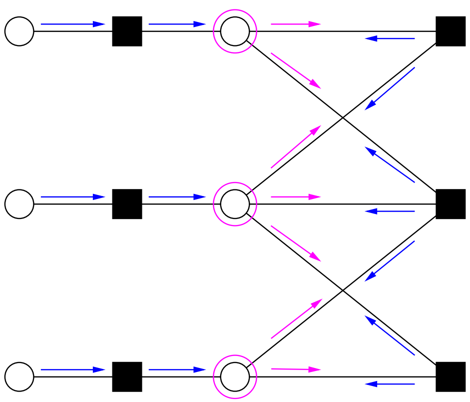

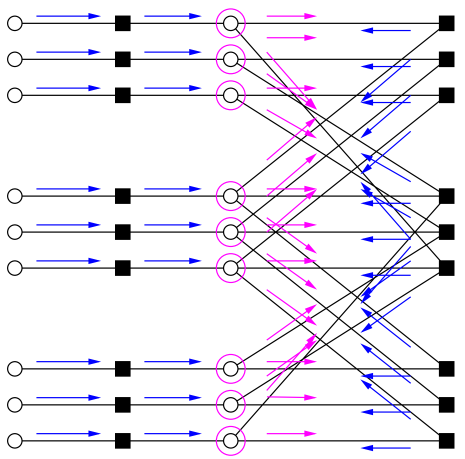

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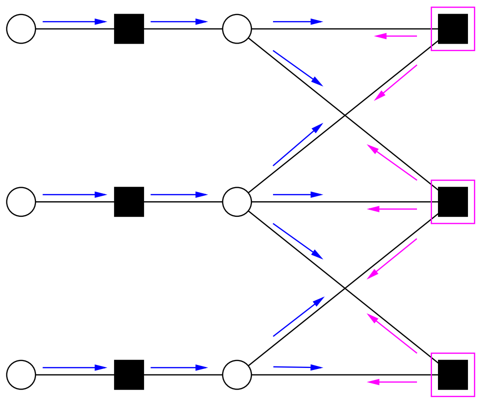

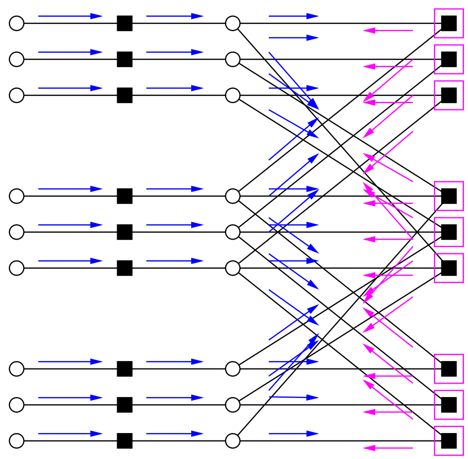

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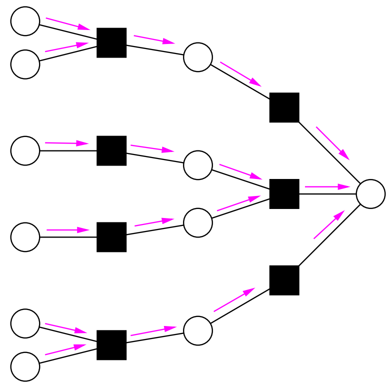

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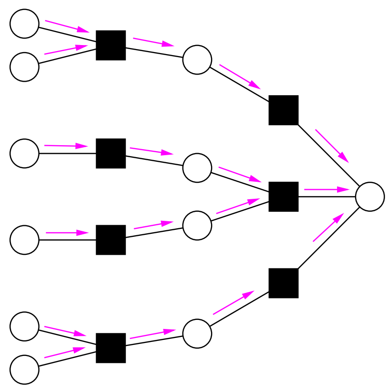

Message-Passing Iterative Decoding

Why do factor graph covers matter?

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all codewords from all covers are also competing to be the best!

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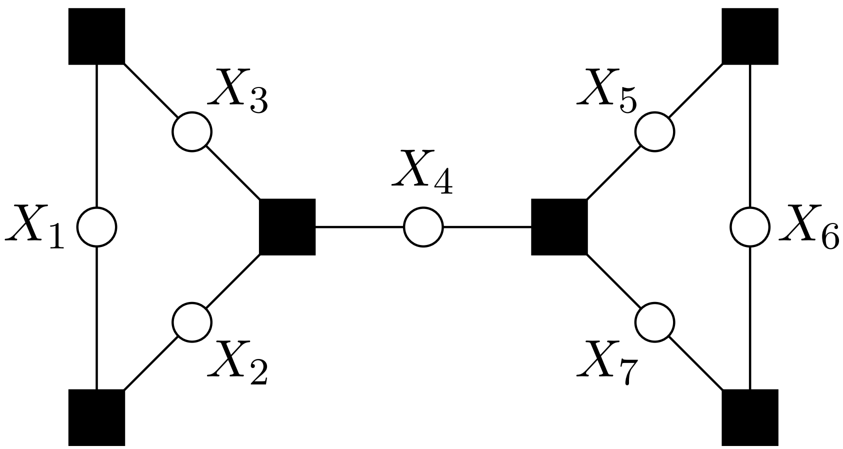
Codewords in graph covers

Codewords in Graph Covers



Base factor/Tanner graph
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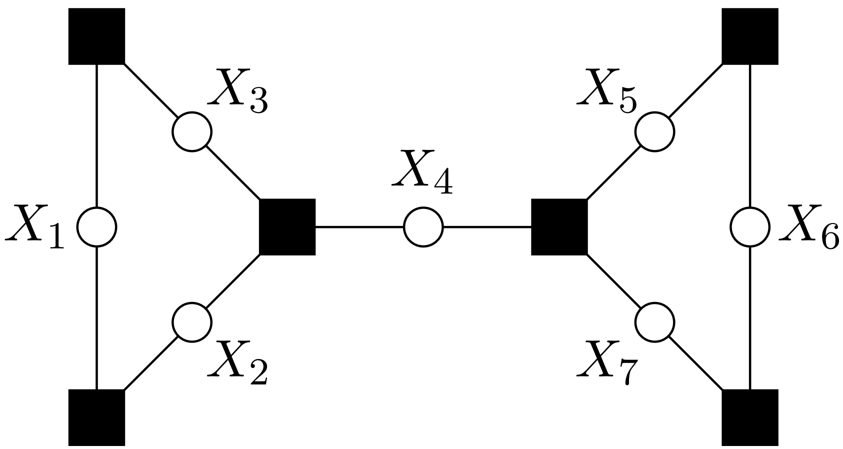

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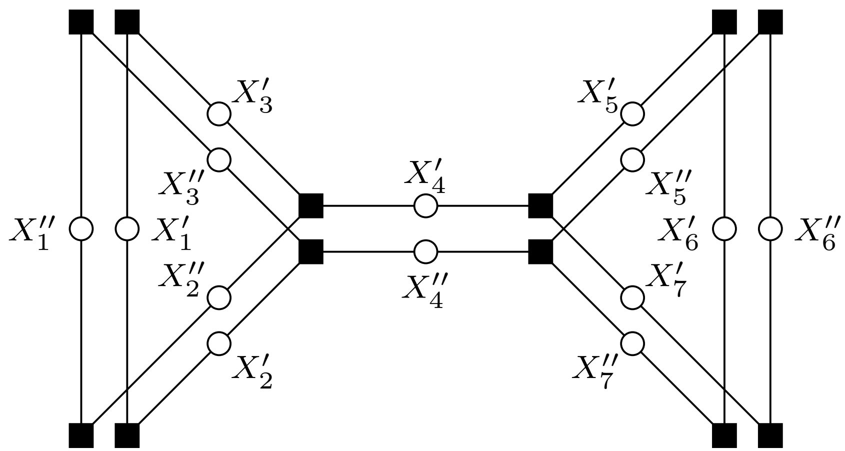

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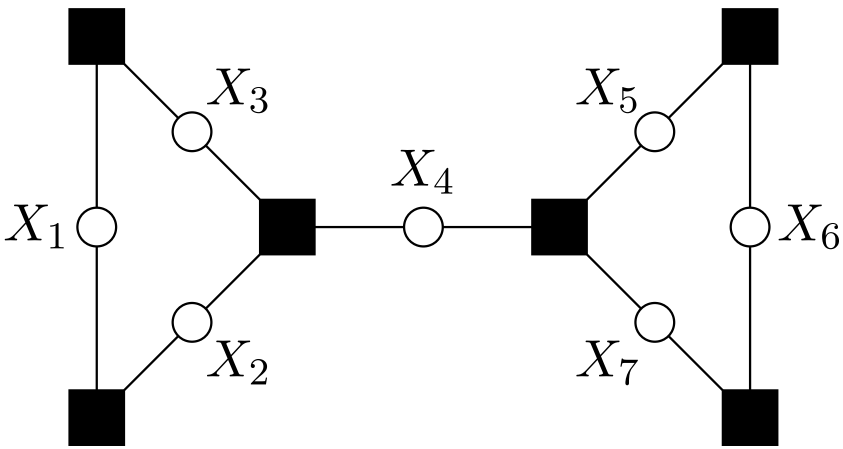

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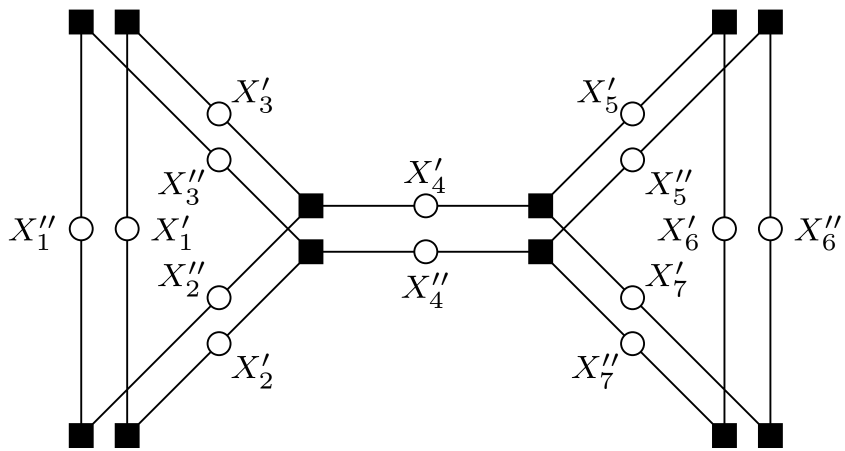
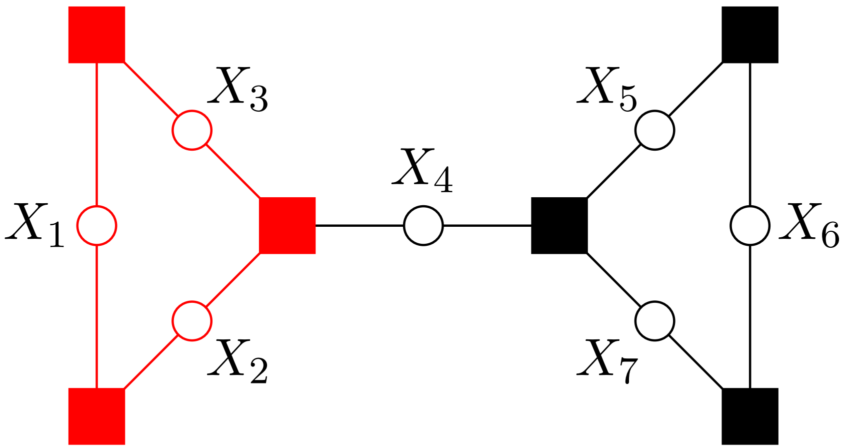


Possible double cover of the base factor graph

Let us study the codes defined by the graph covers of the base Tanner/factor graph.

Codewords in Graph Covers

Obviously, any codeword in the base normal factor graph can be lifted to a codeword in the double cover of the base normal graph.



$\Rightarrow$

$$(1, 1, 1, 0, 0, 0, 0)$$

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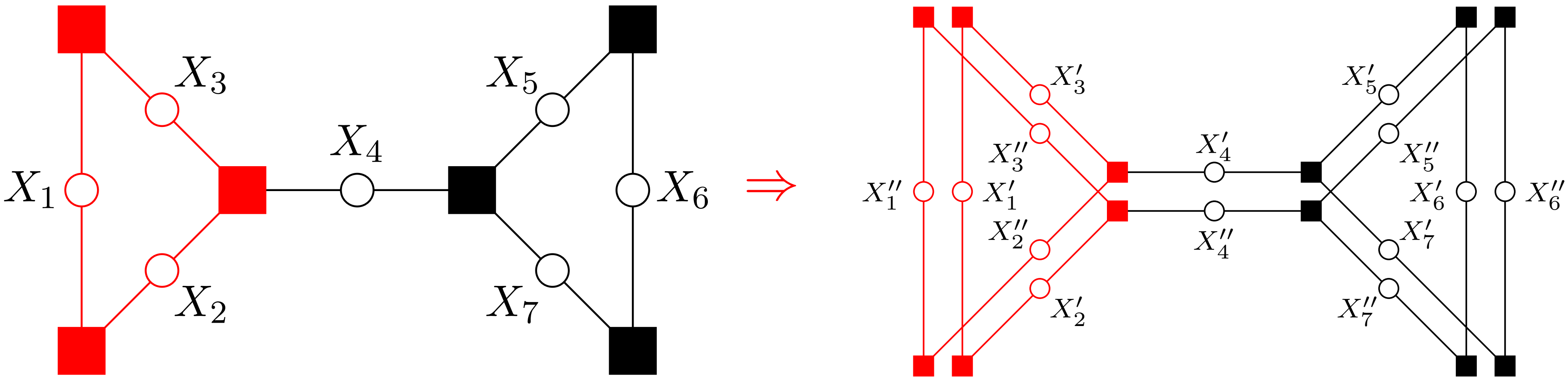

$(1, 1, 1, 0, 0, 0, 0)$ $\quad$ $(1{:}1,\ 1{:}1,\ 1{:}1,\ 0{:}0,\ 0{:}0,\ 0{:}0,\ 0{:}0)$

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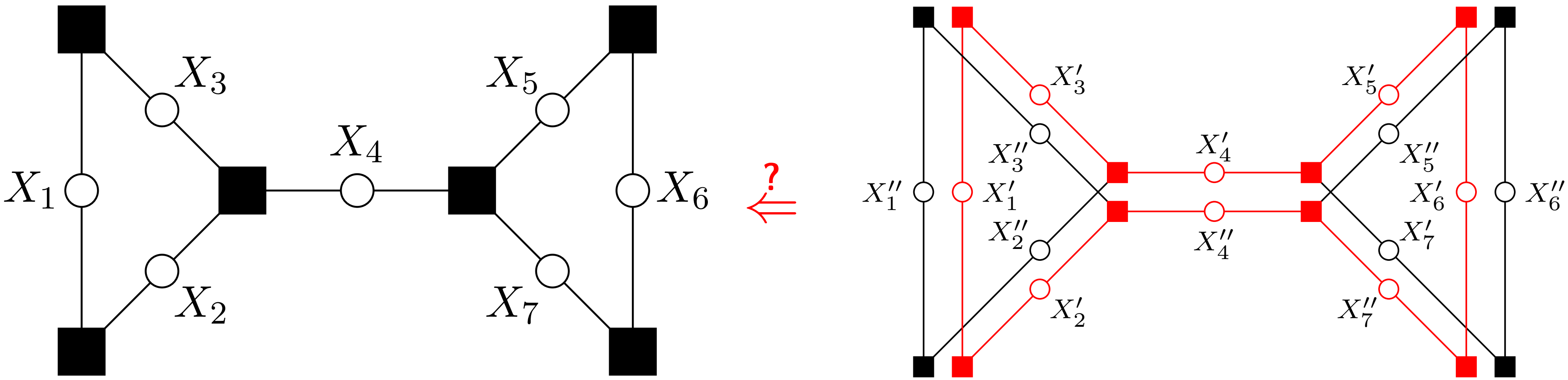
But in the double cover of the base normal factor graph there are also codewords that are not liftings of codewords in the base factor graph!



?

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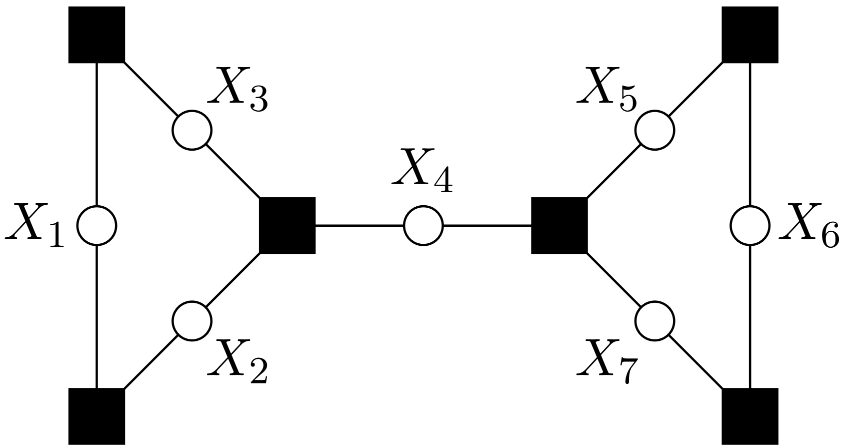

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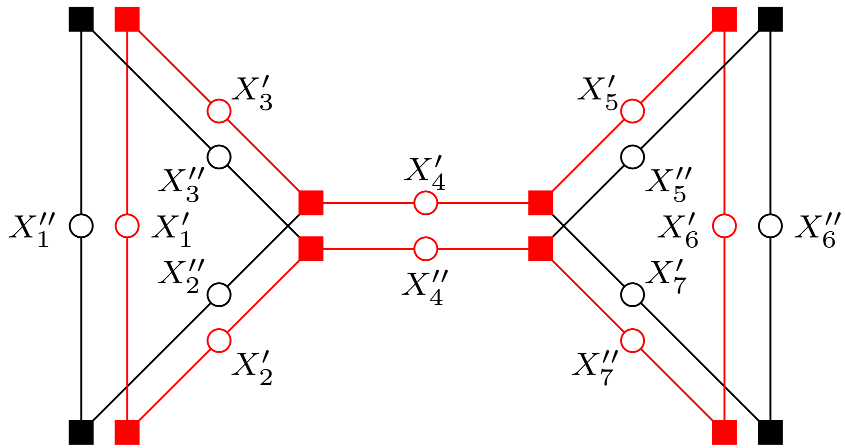

What about

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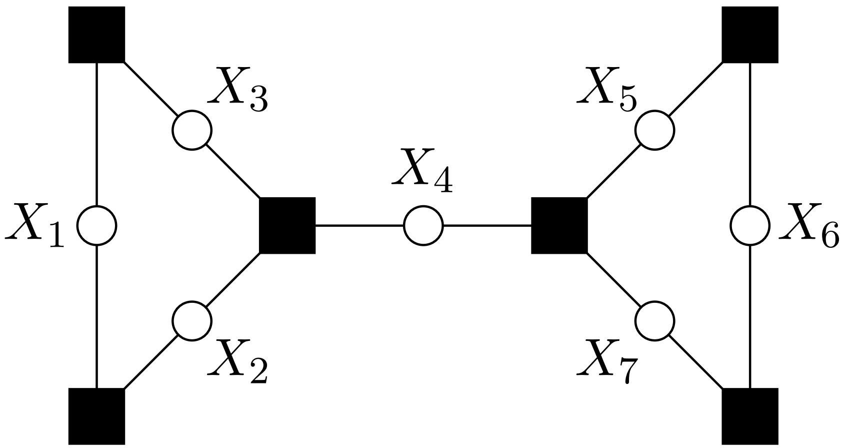

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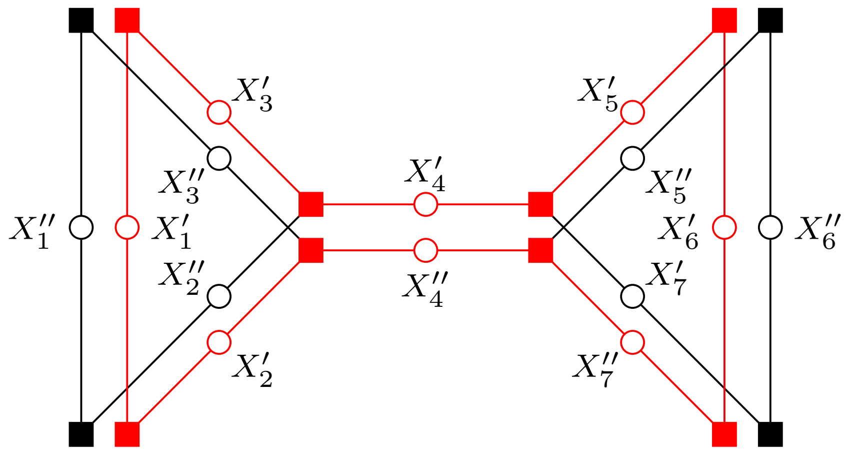

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$\Rightarrow$ We will call such a vector a (graph-cover) pseudo-codeword.

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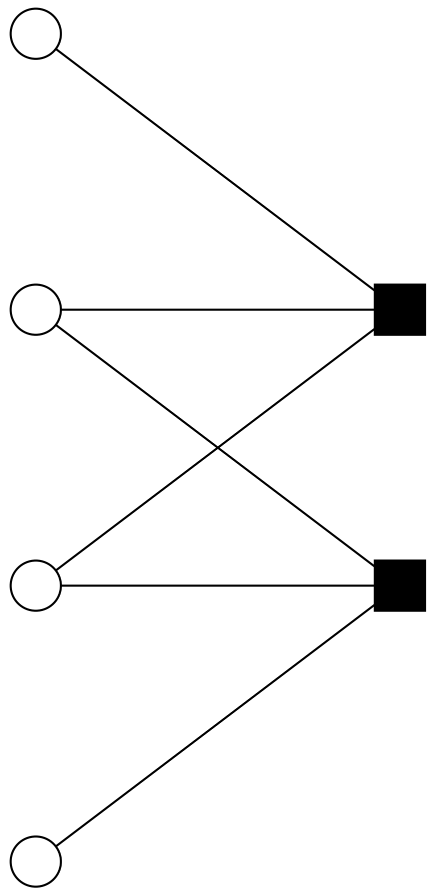

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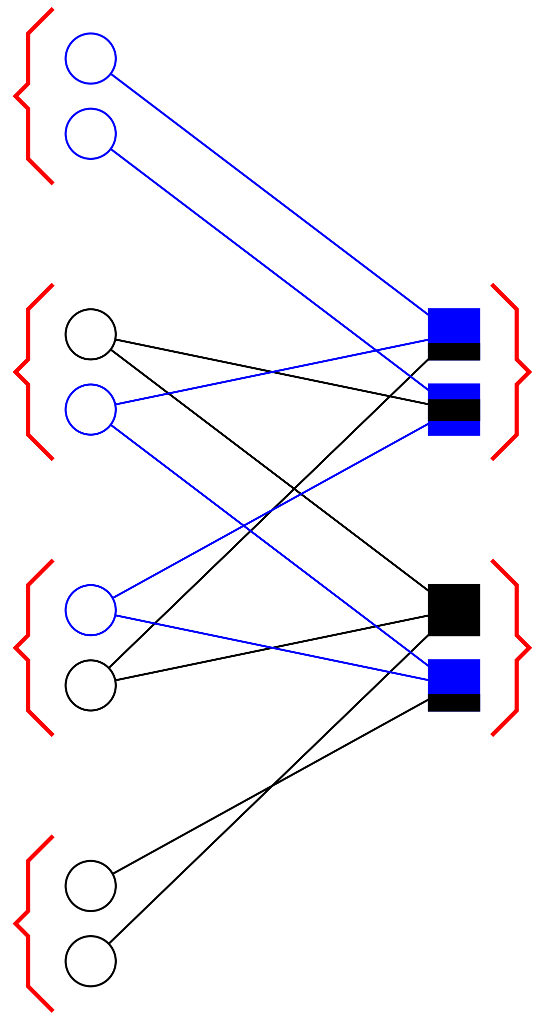

Valid Configurations in Graph Covers

The components of the pseudo-marginal

$$\boldsymbol{\alpha} = \left\{ \{\boldsymbol{\alpha}_i\}_{i \in \mathcal{I}},\ \{\boldsymbol{\alpha}_j\}_{j \in \mathcal{J}} \right\}$$

associated to the valid configuration $\tilde{\mathbf{x}}$ are defined to be

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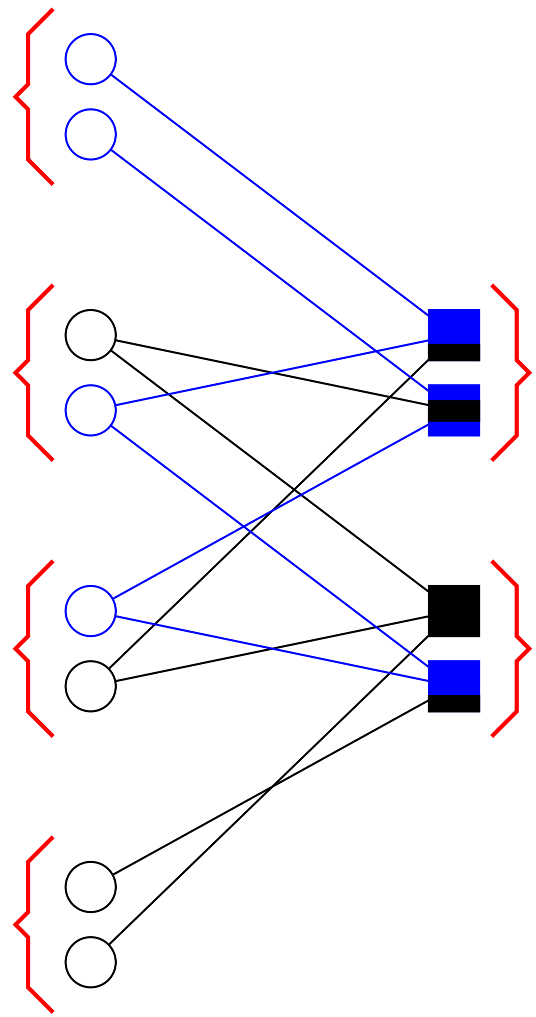

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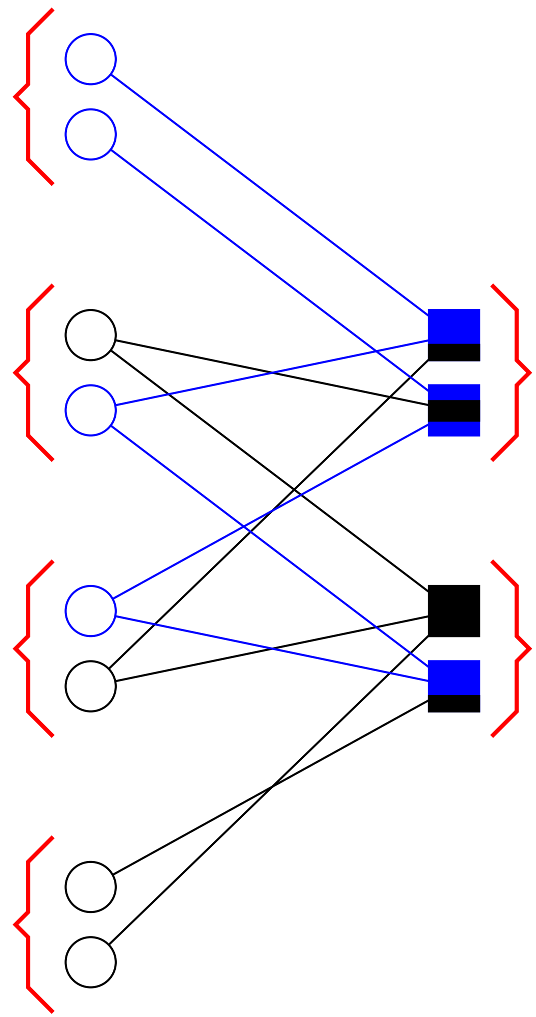
The mapping from any M-fold graph cover to the base graph will be called $\boldsymbol{\varphi}_M$.



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More formally, for any positive integer M, φ_M is the mapping

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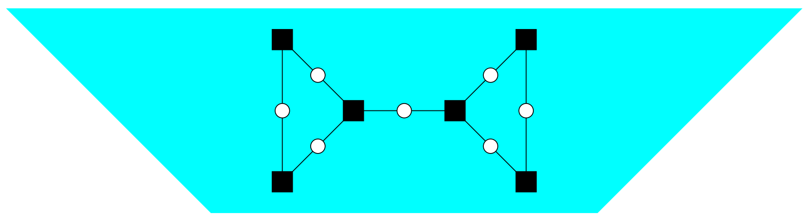
Important: the graph $\widetilde{\mathsf{G}} \in \widetilde{\mathcal{G}}_M$ needs to be included in the tuple $\big(\widetilde{\mathsf{G}}, \widetilde{\mathbf{x}}\big)$, otherwise $\widetilde{\mathbf{x}}$ is not well defined. This is especially crucial once we consider the inverse mapping $\boldsymbol{\varphi}_M^{-1}$.

Counting codewords in graph covers

Graph Cover Hierarchy



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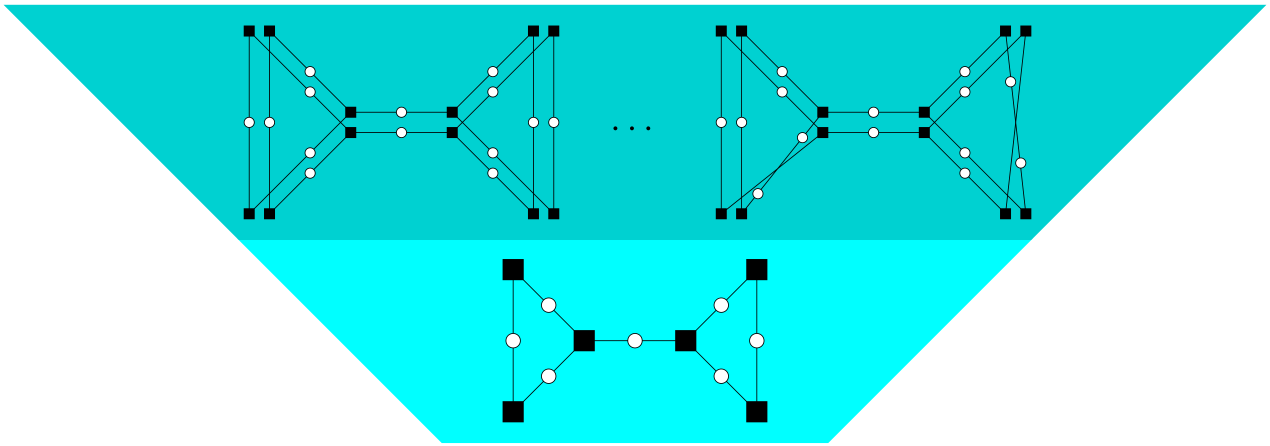

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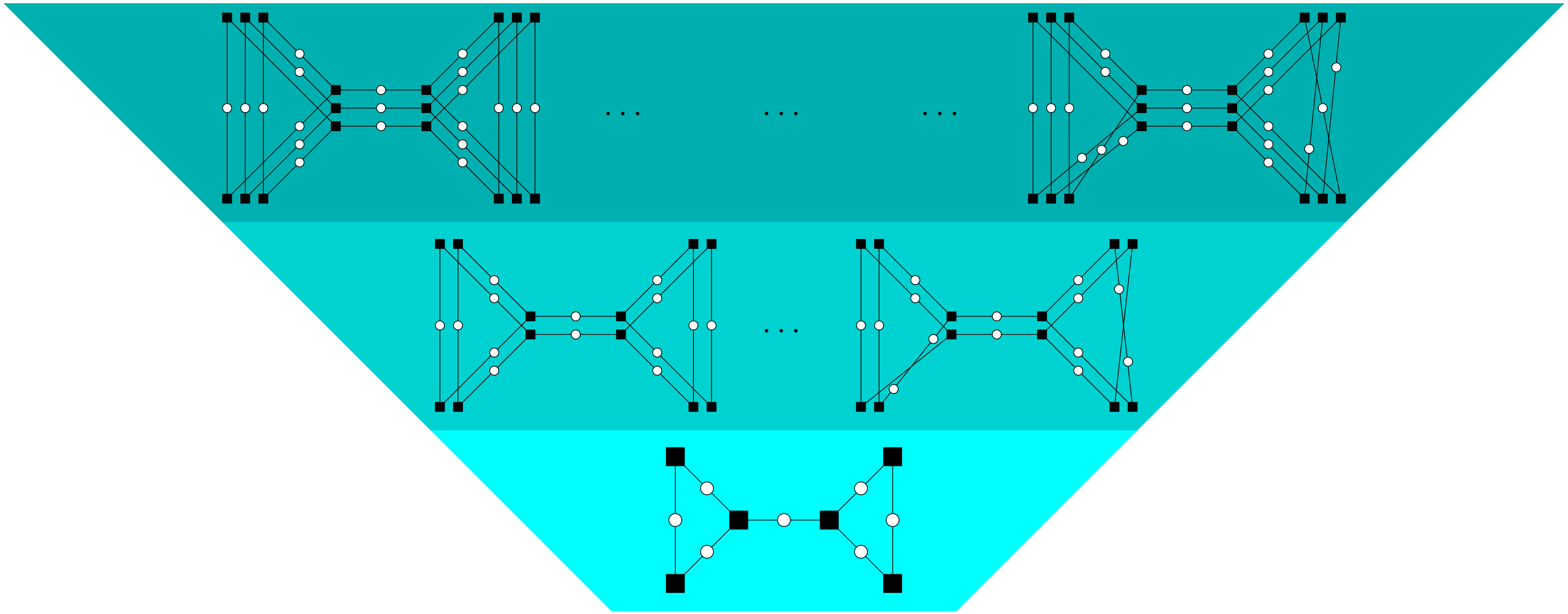

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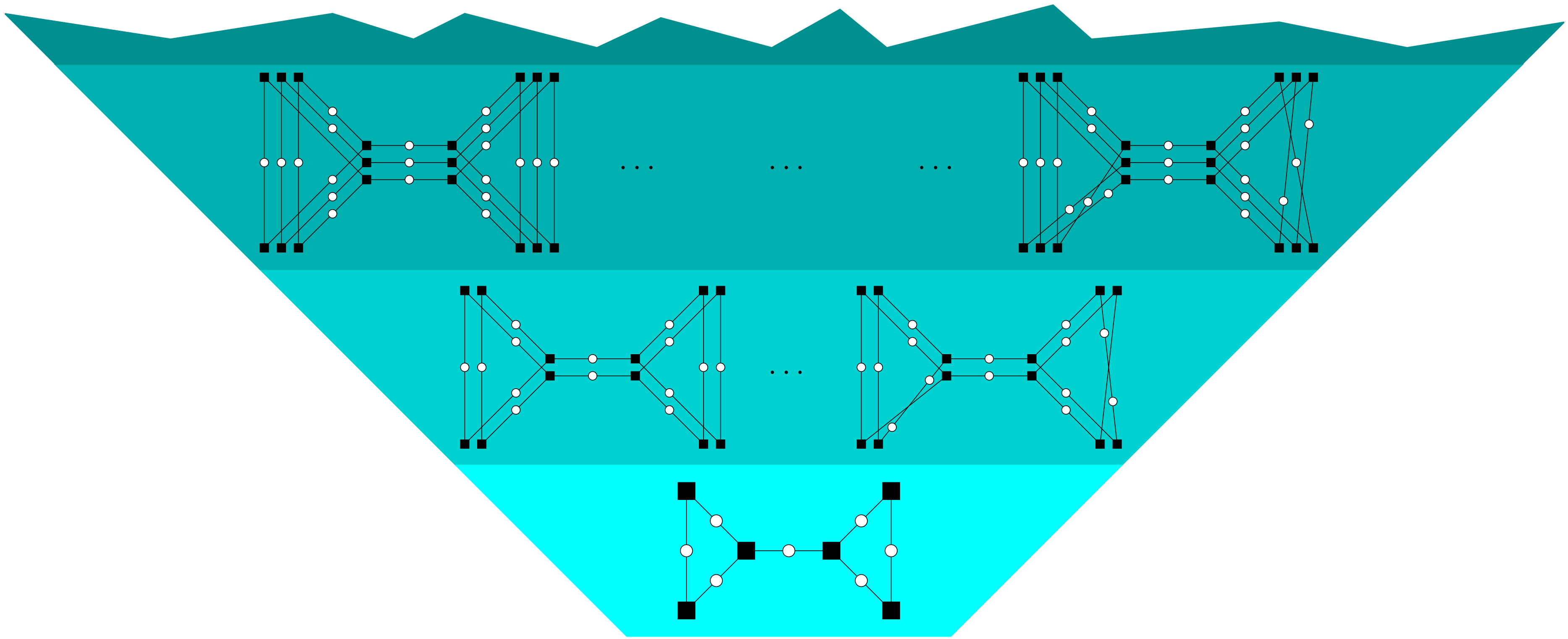

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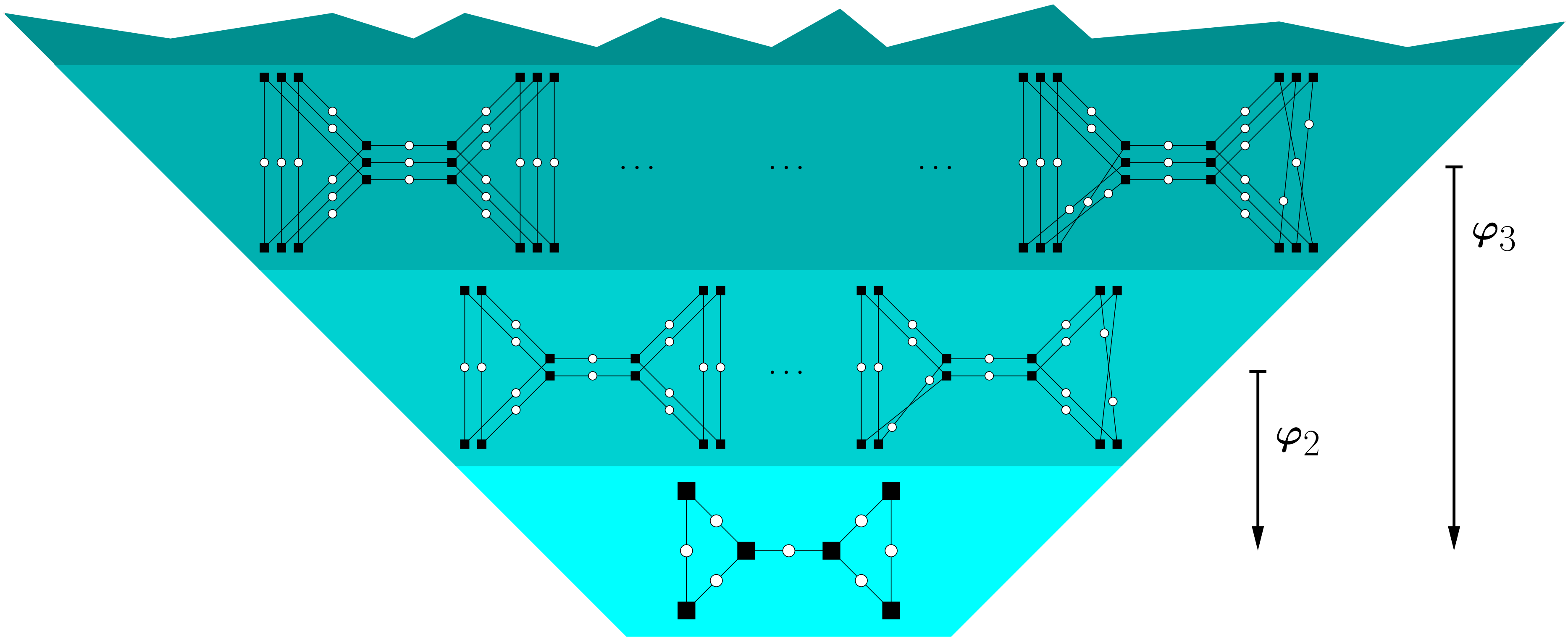

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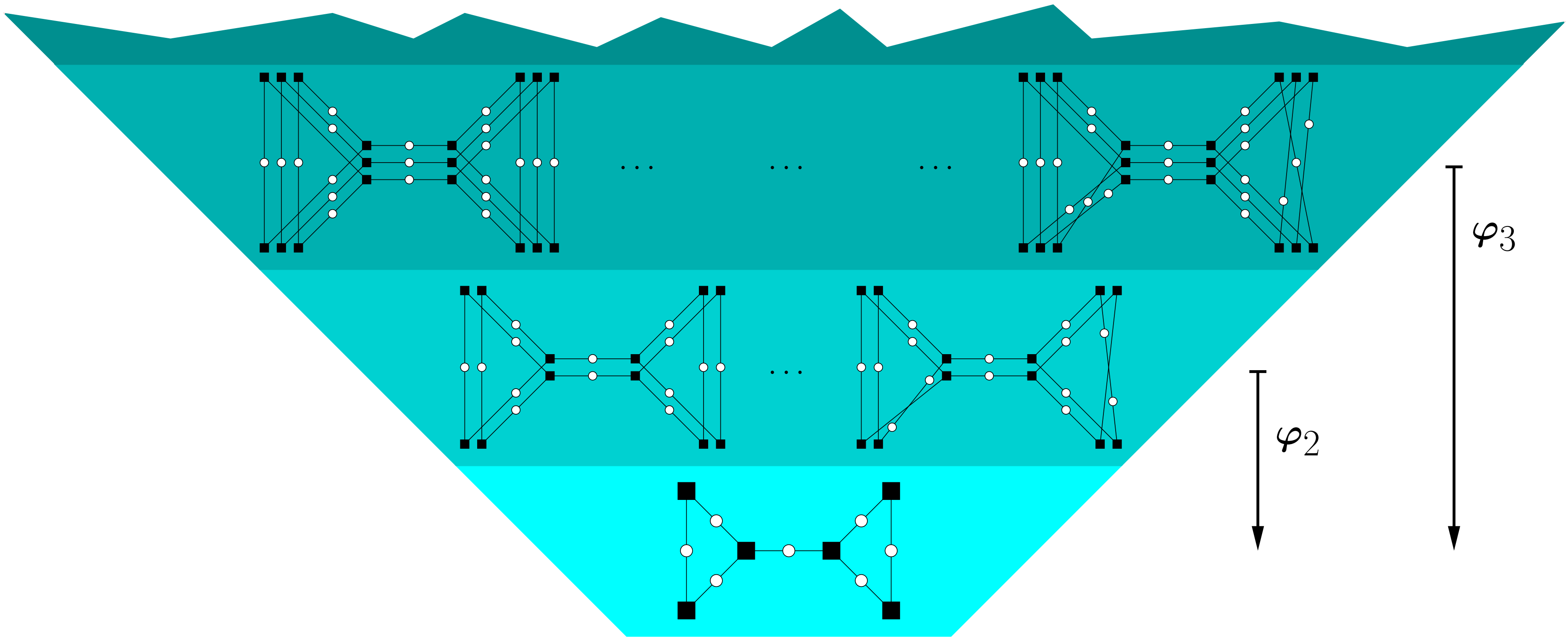

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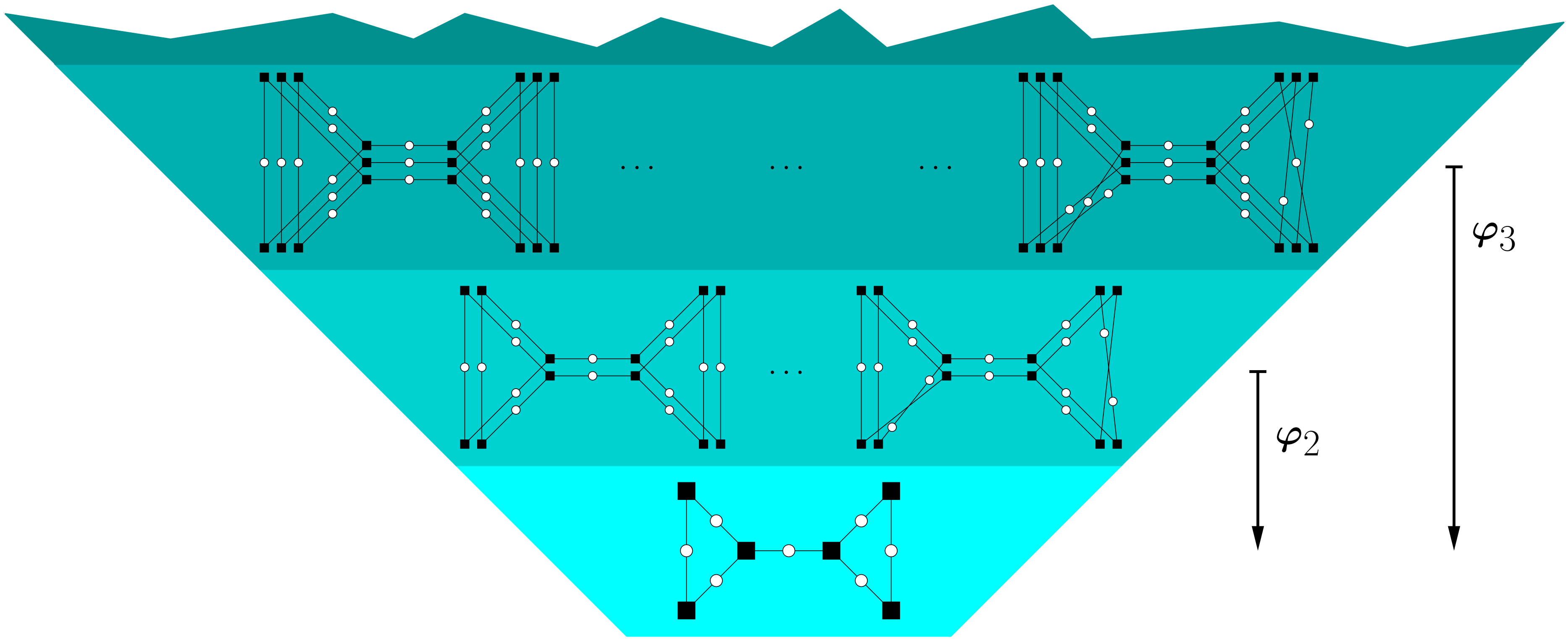

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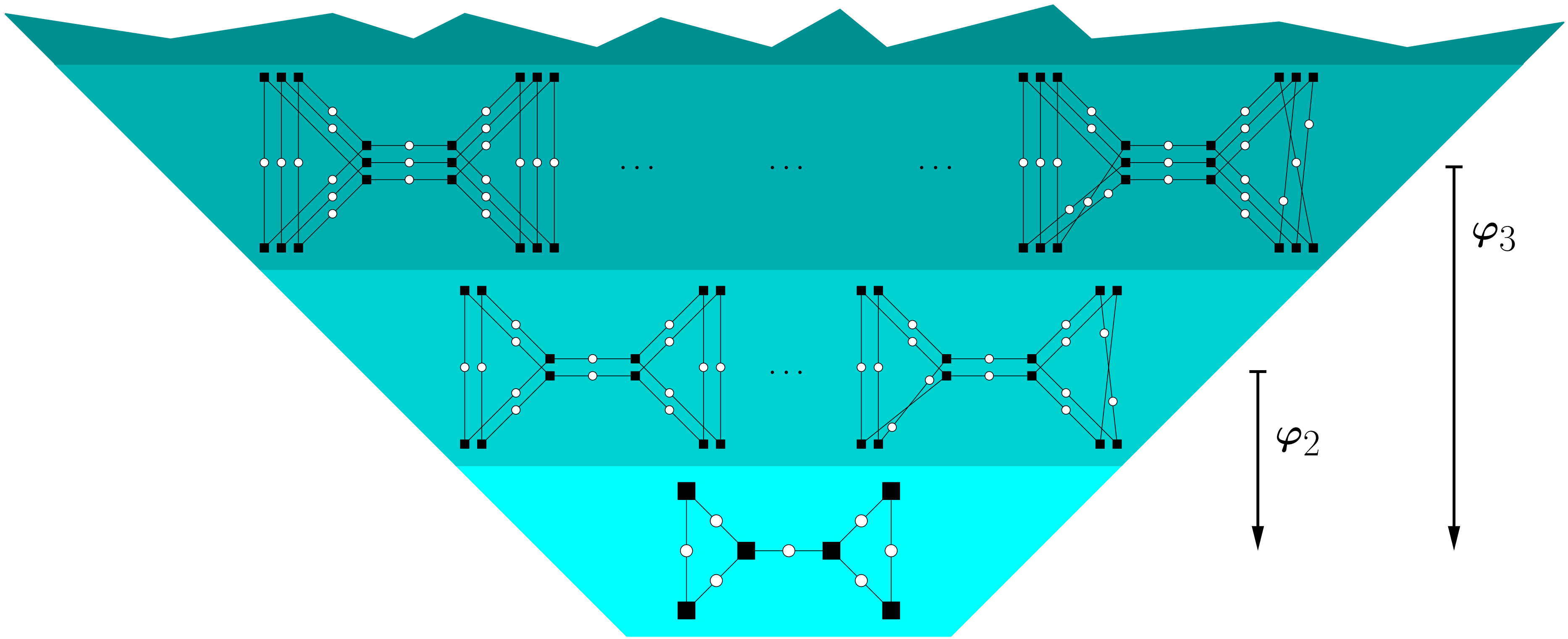

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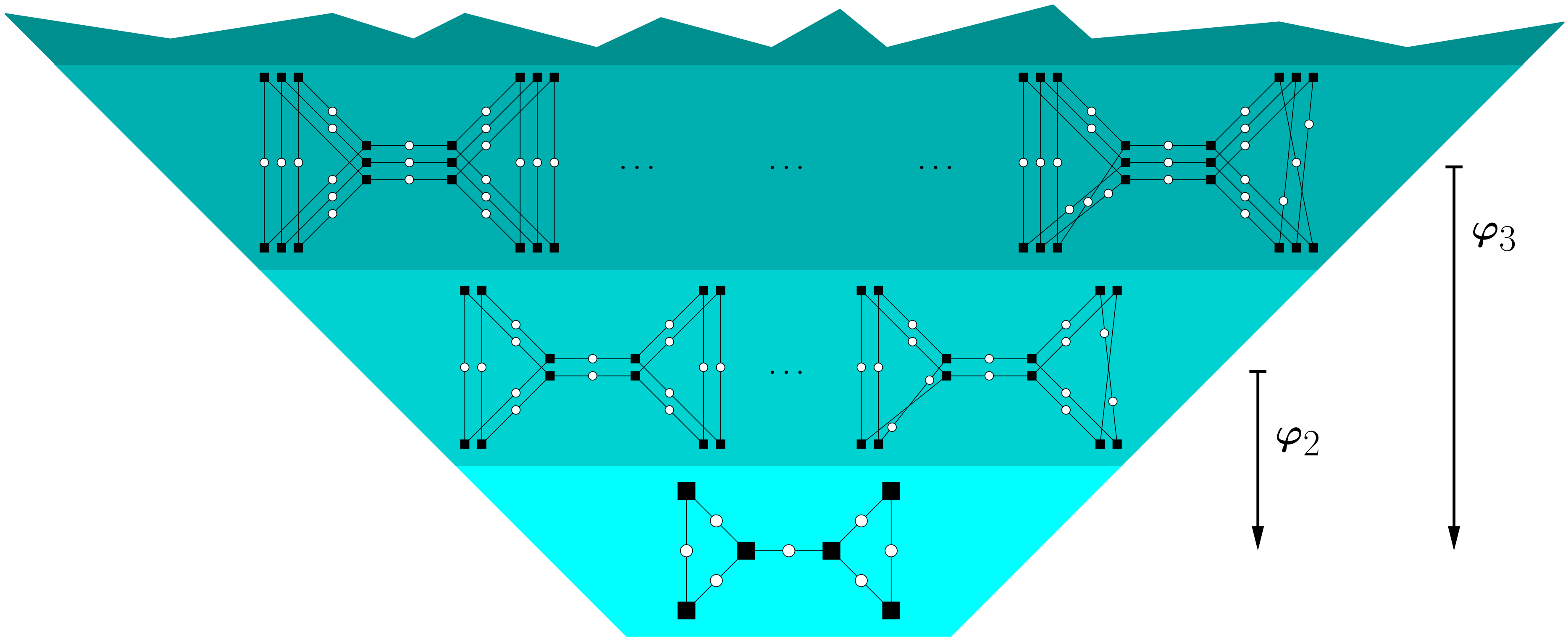

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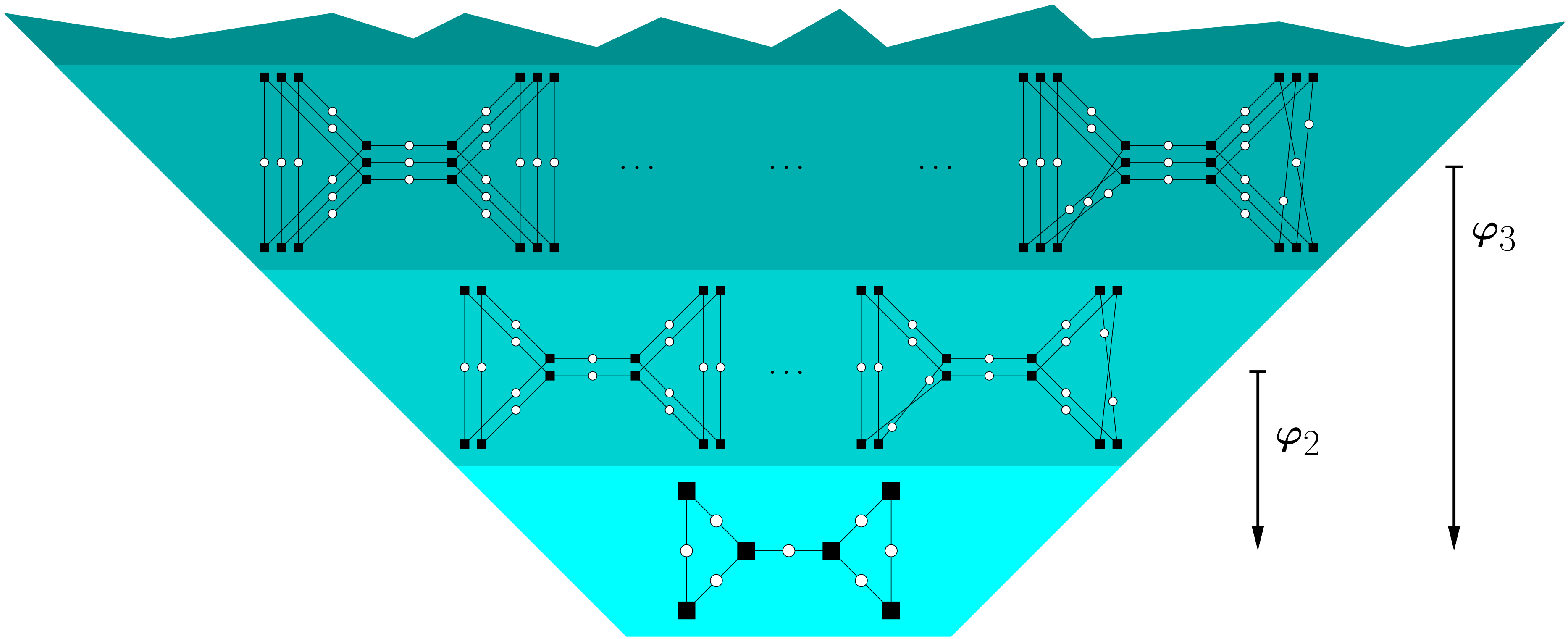

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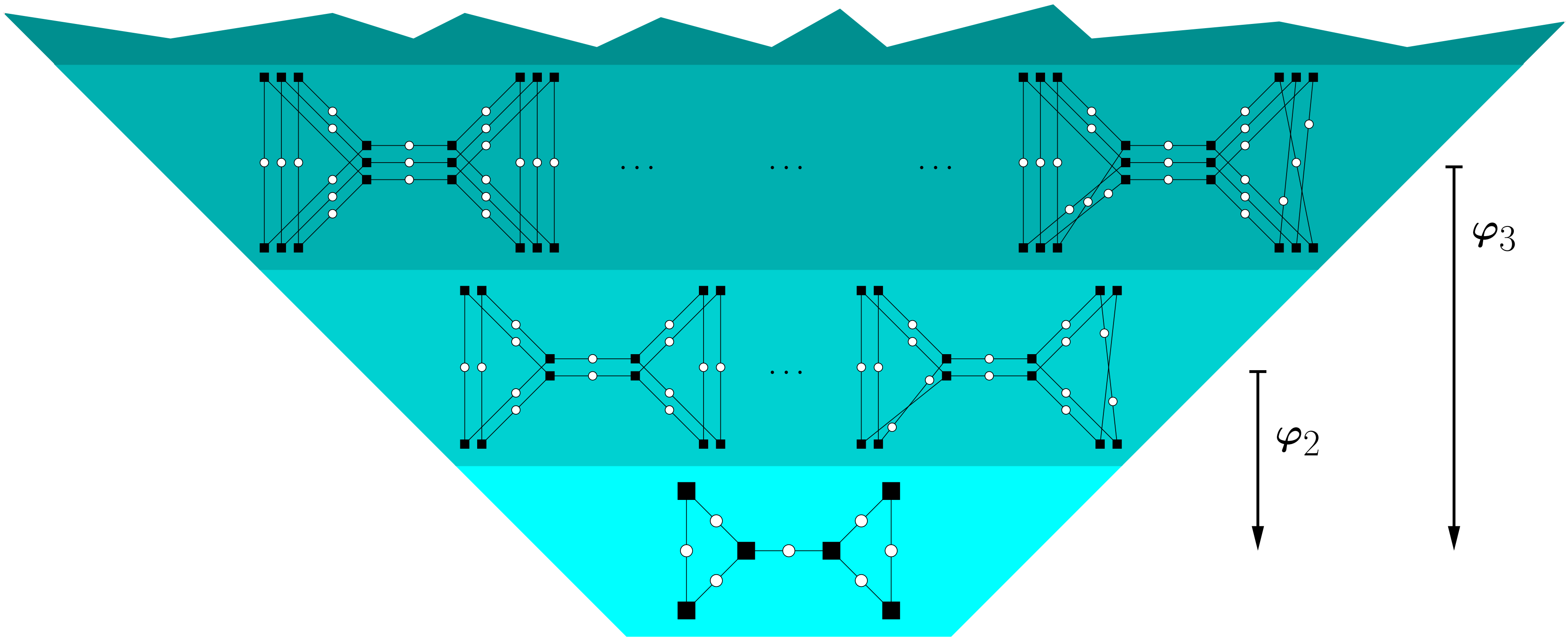

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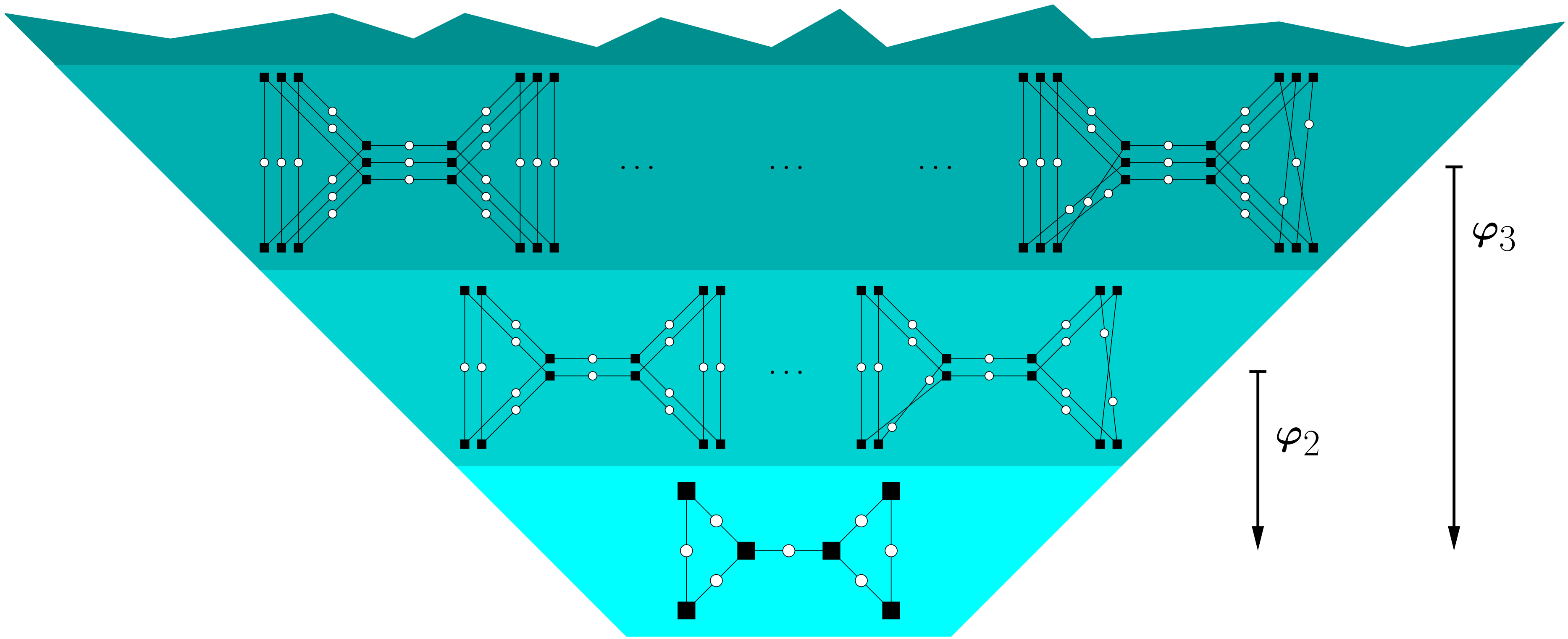

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$$\textbf{Theorem:} \quad \limsup_{M \to \infty} \frac{1}{M} \log \frac{\# \boldsymbol{\varphi}_M^{-1}(\boldsymbol{\alpha})}{\# \widetilde{\mathcal{G}}_M} = H_{\mathrm{Bethe}}(\boldsymbol{\alpha})$$

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- Note: The ratio

$$\frac{\#\boldsymbol{\varphi}_M^{-1}(\boldsymbol{\alpha})}{\#\widetilde{\mathcal{G}}_M}$$

  represents the average number of valid configurations $\tilde{\mathbf{x}}$ per M-fold cover with associated pseudo-marginal $\boldsymbol{\alpha}$. Therefore, $H_{\mathrm{Bethe}}(\boldsymbol{\alpha})$ gives the asymptotic growth rate of that quantity.

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- To the best of our knowledge, the above interpretation of the Bethe entropy cannot be found in the literature (besides the talks that we gave at the 2008 Allerton Conference / 2009 ITA Workshop in San Diego).

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$$\begin{aligned} P(\text{macrostate}) &\propto \#\big\{\text{microstate} : \ \boldsymbol{\varphi}_M(\text{microstate}) = \text{macrostate}\big\} \\ &= \#\boldsymbol{\varphi}_M^{-1}(\text{macrostate}) \\ &\propto \frac{\#\boldsymbol{\varphi}_M^{-1}(\text{macrostate})}{\#\widetilde{\mathcal{G}}_M} \end{aligned}$$

"Micro-/Macrostate Reinterpretation" of the Theorem by Yedidia et al.

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Fixed Points of the SPA

Theorem (Yedidia/Freeman/Weiss, 2000)

Fixed points of the SPA correspond to stationary points of the Variational Bethe free energy (VBFE).

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Re-interpretation in terms of graph covers:

A fixed point of the SPA corresponds to a macrostate $\boldsymbol{\alpha}$, i.e., a pseudo-marginal $\boldsymbol{\alpha}$, that is a *stationary point* of

$$P(\boldsymbol{\alpha}) \;\propto\; \exp\Big(-M \cdot \langle \boldsymbol{\alpha}, \boldsymbol{\lambda} \rangle\Big) \cdot \#\varphi^{-1}(\boldsymbol{\alpha})$$

when M goes to infinity.

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The Transient Part of the SPA

Static vs. Dynamic Setup

Symbols: σ: microstate, Σ: macrostate.

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Dynamic setup:

$$P\big(\Sigma(t+\Delta t) \,\big|\, \sigma(t)\big)$$
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"Better" dynamic setup: $\boxed{\text{will model the transient part of the SPA}}$

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Graph-Dynamical Systems

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Graph-dynamical system (e.g., [Prisner:95]):

- Let Γ be a set of graphs.
- Let Ψ be some (possibly random) mapping from Γ to Γ.
- Because the domain and the range of Ψ are equal, it makes sense to study the repeated application of the mapping Ψ:

$$\Gamma \quad \xrightarrow{\Psi} \quad \Gamma \quad \xrightarrow{\Psi} \quad \cdots \quad \xrightarrow{\Psi} \quad \Gamma$$

Review

(of the setup used in the re-interpretation of f.p.s of the SPA)

- Set of microstates

$$\triangleq \left((\widetilde{\mathsf{G}}, \widetilde{\mathbf{x}}) \;\middle|\; \widetilde{\mathsf{G}} \in \widetilde{\mathcal{G}}_M, \; \widetilde{\mathbf{x}} \text{ is a valid configuration in } \widetilde{\mathcal{G}}_M \right)$$

- Mapping $\boldsymbol{\varphi}_M$

$$\text{maps} \quad (\widetilde{\mathsf{G}}, \widetilde{\mathbf{x}}) \quad \text{to} \quad \boldsymbol{\omega}(\widetilde{\mathbf{x}})$$

- Set of macrostates

$$\triangleq \boldsymbol{\varphi}_M(\text{set of microstates})$$

Corresponding Setup for the Transient Part of the SPA

- Set of microstates

???

- Mapping φ_M

???

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Corresponding Setup for the Transient Part of the SPA

- Set of microstates

$$???$$

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$$???$$

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$$???$$

Note: $\Gamma =$ set of M-covers of G and valid configurations therein is obviously not sufficient.

Corresponding Setup for the Transient Part of the SPA

- Set of microstates

  $\Rightarrow \ \Gamma =$ set of what we call colored hypergraph M-cover

  or colored twisted M-cover

- Mapping φ_M

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  $\Rightarrow$ $\Gamma =$ set of what we call colored hypergraph M-cover
  or colored twisted M-cover

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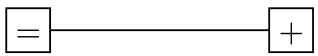
- Set of macrostates

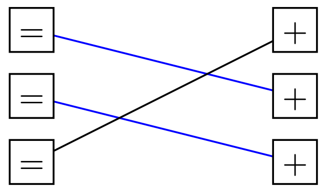
  set of all possible marginals on the LHS function nodes
  $\times$ set of all possible marginals on the RHS function nodes

Comment on Microstates



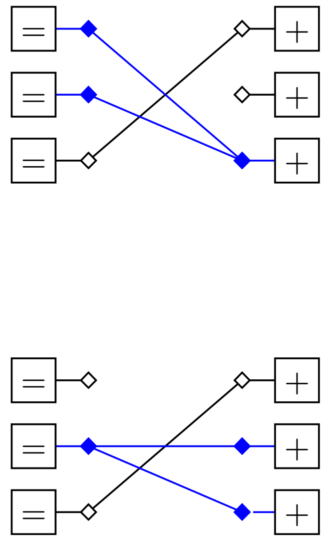
edge

in FFG



corrsponding edges

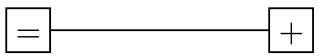
in some colored 3-cover



corresponding edges

in colored hypergraph 3-cover

Comment on Microstates



edge

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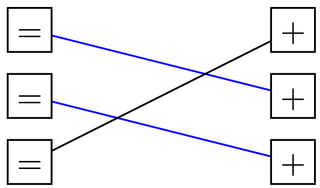
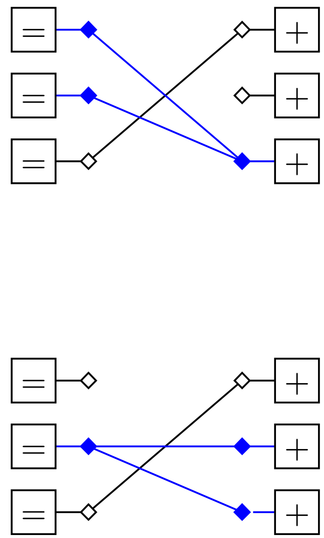


corrsponding edges

in some colored 3-cover

LHS and RHS marginals must match



corresponding edges

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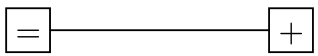

edge

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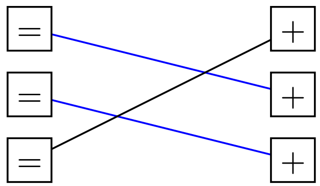

corrsponding edges

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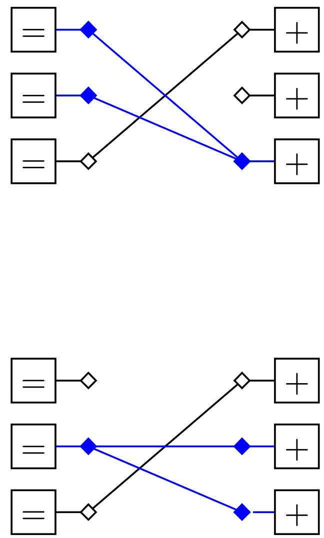

corresponding edges

in colored hypergraph 3-cover

LHS and RHS marginals
do not have to match

Comment on Macrostates

$$
\begin{tikzcd}
\overleftarrow{\boldsymbol{\mu}}(0) \arrow[r] \arrow[d] & \overrightarrow{\boldsymbol{\mu}}(0.5) \arrow[r] \arrow[d] & \overleftarrow{\boldsymbol{\mu}}(1) \arrow[r] \arrow[d] & \overrightarrow{\boldsymbol{\mu}}(1.5) \arrow[r] \arrow[d] & \overleftarrow{\boldsymbol{\mu}}(2) \arrow[r] \arrow[d] & {} \\
\boldsymbol{\eta}_{\mathrm{left}}(0) & \boldsymbol{\eta}_{\mathrm{right}}(0.5) & \boldsymbol{\eta}_{\mathrm{left}}(1) & \boldsymbol{\eta}_{\mathrm{right}}(1.5) & \boldsymbol{\eta}_{\mathrm{left}}(2) & .
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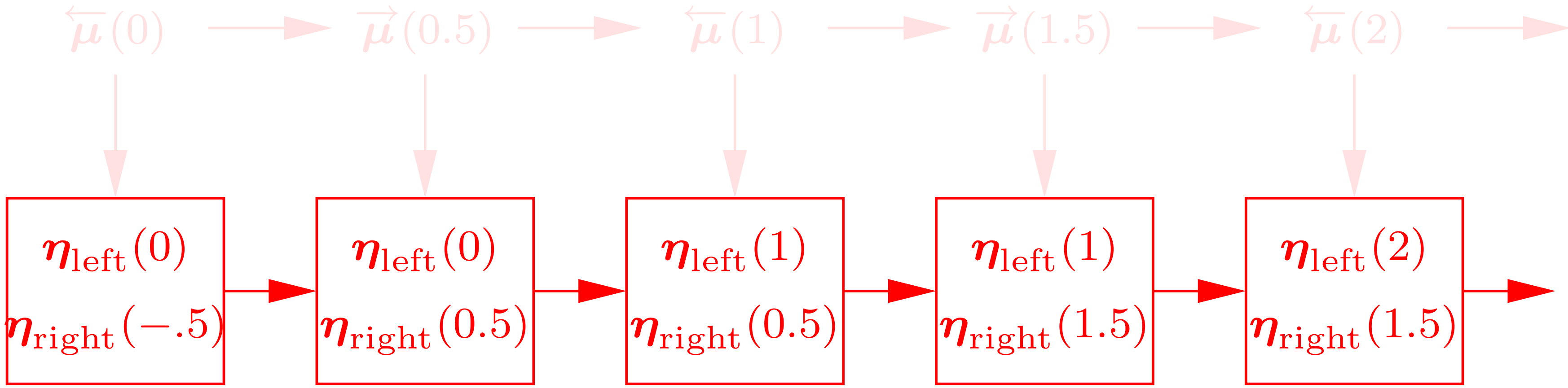

.

$\Rightarrow$ This can be considered as a "message-free version of the SPA".

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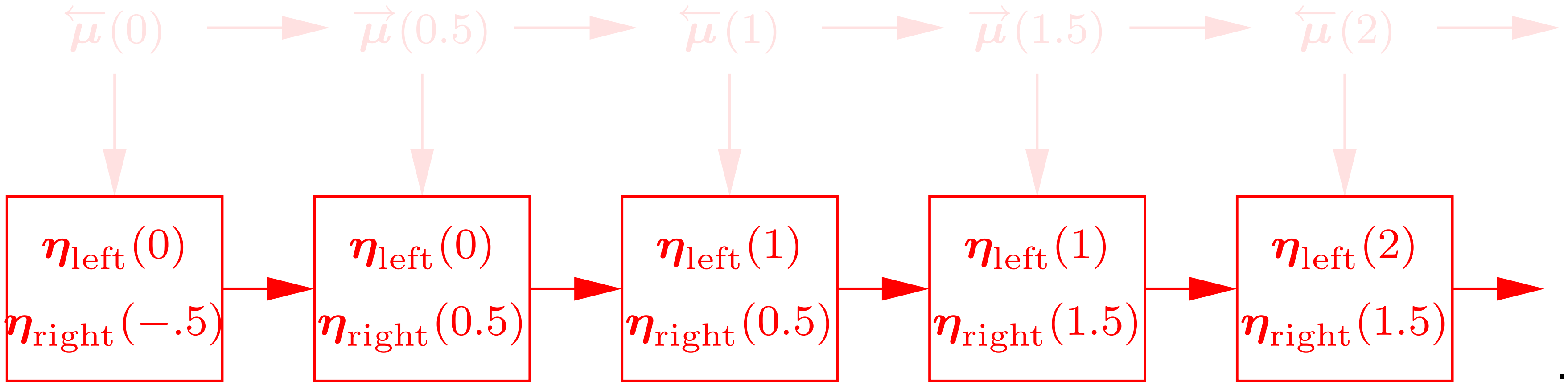

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Cf. “Message-free version of belief-propagation”
in [Wainwright/Jaakkola/Willsky, 2003].

Bethe Entropy and Weight Spectra

Induced Bethe Entropy

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For any $\boldsymbol{\omega} \in \mathcal{P}(\mathbf{H})$, define the induced Bethe entropy to be

$$H_{\text{Bethe}}(\boldsymbol{\omega}) \triangleq H_{\text{Bethe}}(\boldsymbol{\alpha})\Big|_{\boldsymbol{\alpha}=\boldsymbol{\Psi}_{\text{BME}}(\boldsymbol{\omega})} ,$$

where $\boldsymbol{\Psi}_{\text{BME}}(\boldsymbol{\omega})$ is the Bethe max-entropy pseudo-marginal $\boldsymbol{\alpha} \in \mathcal{L}$ among all the pseudo-marginals in $\mathcal{L}$ that correspond to $\boldsymbol{\omega}$.

Evaluating the Bethe Free Entropy Along the Cube Diagonal

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- Take some finite-length (j, k)-regular LDPC code of length n.

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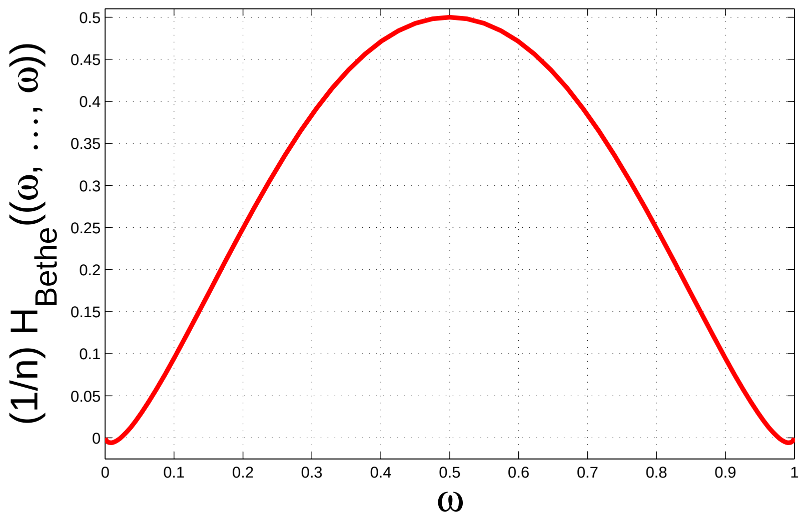

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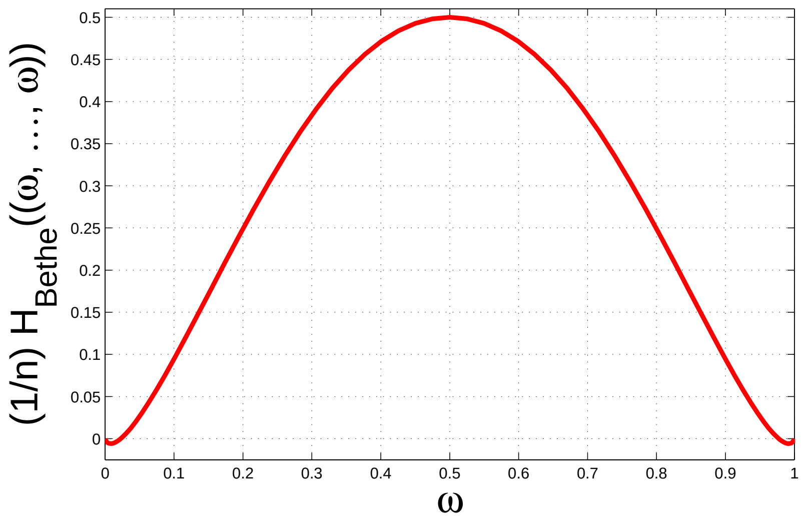

- This function happens to equal the exponent of the asymptotic average Hamming weight distribution for Gallager's ensemble of (j, k)-regular LDPC codes!

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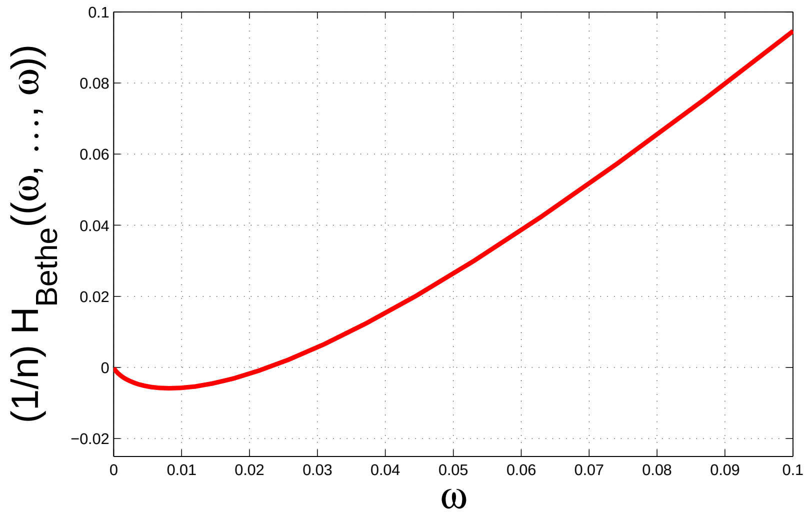

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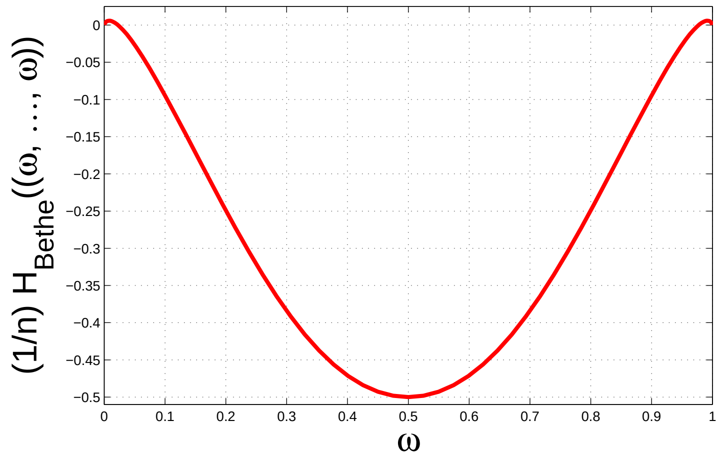

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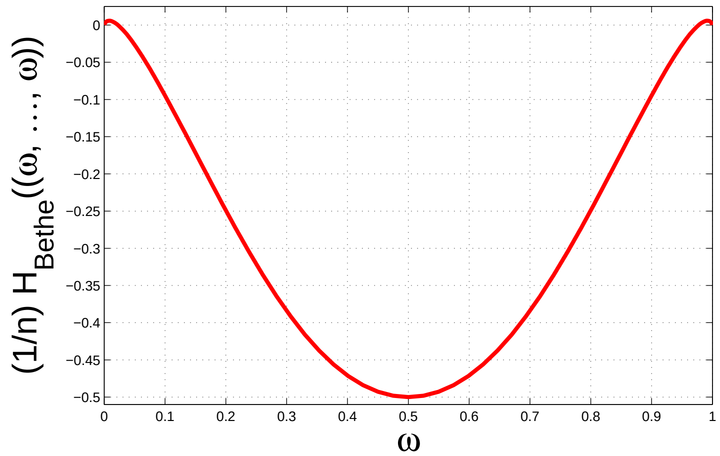

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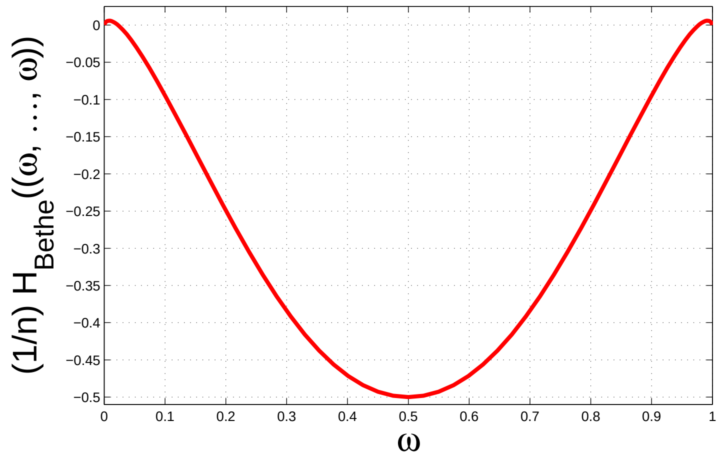

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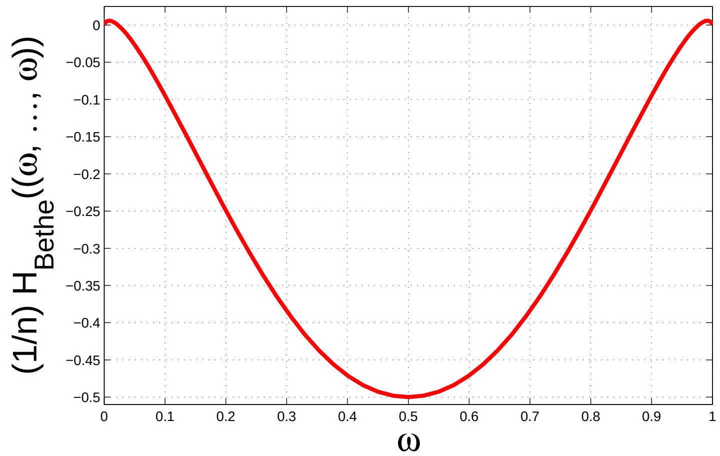
Evaluating the Bethe Free Entropy Along the Cube Diagonal

- Let's look at $-\frac{1}{n} H_{\mathrm{Bethe}}\big((\omega, \ldots, \omega)\big)$.



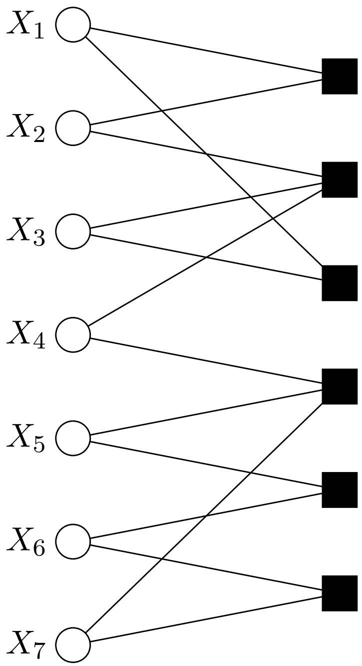
- Remember that $F_{\mathrm{Bethe}}(\boldsymbol{\omega}) = U_{\mathrm{Bethe}}(\boldsymbol{\omega}) - H_{\mathrm{Bethe}}(\boldsymbol{\omega})$.
- Remember that $U_{\mathrm{Bethe}}(\boldsymbol{\omega})$ is linear in $\boldsymbol{\omega}$.
- Therefore, we see that for a finite-length code from an ensemble with asymptotically linearly growing minimum Hamming distance, $F_{\mathrm{Bethe}}(\boldsymbol{\omega})$ is not a convex function of $\boldsymbol{\omega}$.

Bethe Entropy and the Edge Zeta Function

Tanner/Factor Graph of a Cycle Code

Cycle codes are codes which have a Tanner/factor graph where all bit nodes have degree two. (Equivalently, the parity-check matrix has two ones per column.)

Example:



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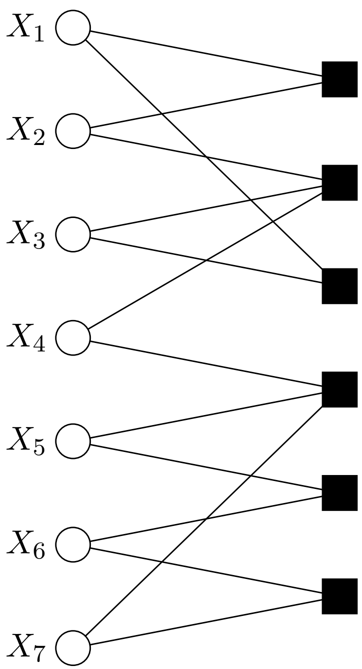
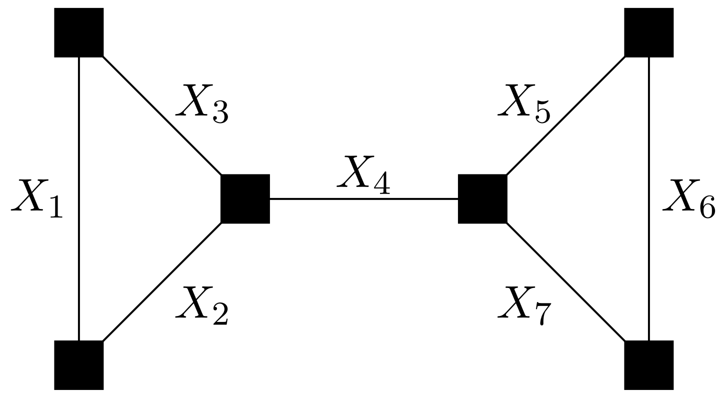


Tanner/factor graph

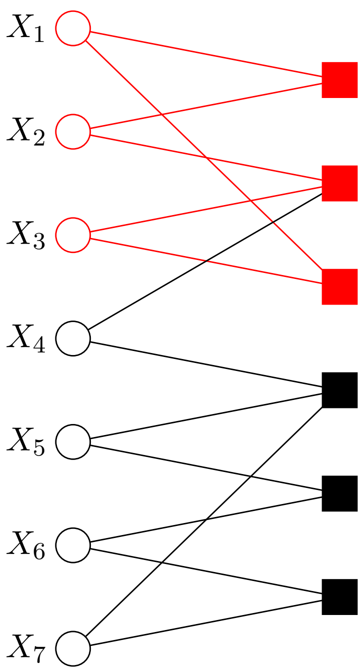


Corresponding normal factor graph

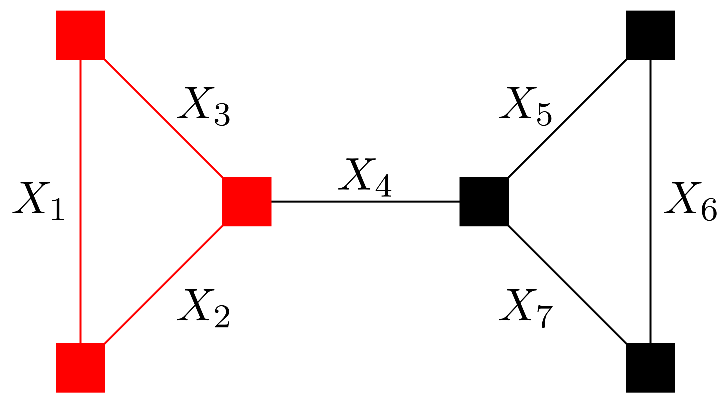
Tanner/Factor Graph of a Cycle Code

Cycle codes are called cycle codes because codewords correspond to simple cycles (or to the symmetric difference set of simple cycles) in the Tanner/factor graph.

Example:



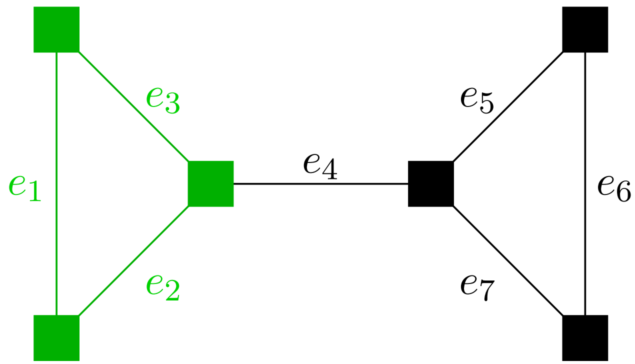
Tanner/factor graph



Corresponding normal factor graph

The Edge Zeta Function of a Graph

Definition (Hashimoto, see also Stark/Terras):



Here: $\Gamma = (e_1, e_2, e_3)$

Let Γ be a path in a graph X with edge-set E; write

$$\Gamma = (e_{i_1}, \ldots, e_{i_k})$$

to indicate that Γ begins with the edge e_{i_1} and ends with the edge e_{i_k}.

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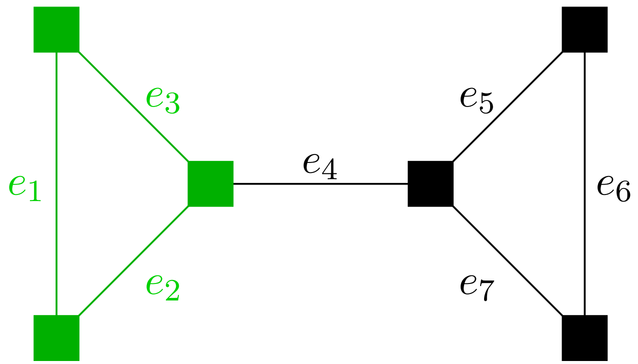

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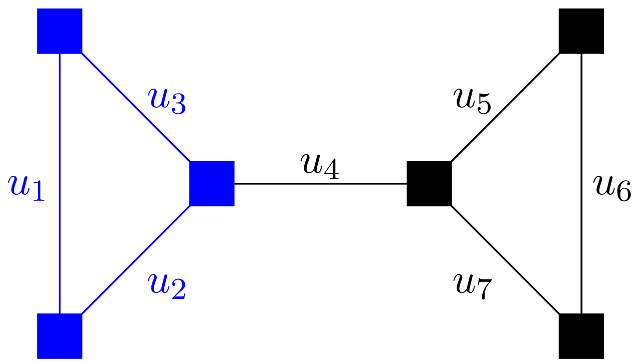

Here: $g(\Gamma) = u_1 u_2 u_3$

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to indicate that Γ begins with the edge e_{i_1} and ends with the edge e_{i_k}.

The monomial of Γ is given by

$$g(\Gamma) \triangleq u_{i_1} \cdots u_{i_k},$$

where the u_i's are indeterminates.

The Edge Zeta Function of a Graph

Definition (Hashimoto, see also Stark/Terras):
The edge zeta function of X is defined to be the power series

$$\zeta_X(u_1, \ldots, u_n) \in \mathbb{Z}[[u_1, \ldots, u_n]]$$

given by

$$\zeta_X(u_1, \ldots, u_n) = \prod_{[\Gamma] \in A(X)} \frac{1}{1 - g(\Gamma)},$$

where $A(X)$ is the collection of equivalence classes of backtrackless, tailless, primitive cycles in X.
Note: unless X contains only one cycle, the set $A(X)$ will be countably infinite.

The Edge Zeta Function of a Graph

Theorem (Bass):

- The edge zeta function $\zeta_X(u_1, \ldots, u_n)$ is a rational function.
- More precisely, for any directed graph $\vec{X}$ of X, we have

$$\zeta_X(u_1, \ldots, u_n) = \frac{1}{\det\left(\mathbf{I} - \mathbf{U}\mathbf{M}(\vec{X})\right)} = \frac{1}{\det\left(\mathbf{I} - \mathbf{M}(\vec{X})\mathbf{U}\right)}$$

where

- $\mathbf{I}$ is the identity matrix of size $2n$,
- $\mathbf{U} = \mathrm{diag}(u_1, \ldots, u_n, u_1, \ldots, u_n)$ is a diagonal matrix of indeterminants.
- $\mathbf{M}(\vec{X})$ is a $2n \times 2n$ matrix derived from some directed graph version $\vec{X}$ of X.

Relationship Pseudo-Codewords and Edge Zeta Function (Part 1: Theorem)

Theorem:

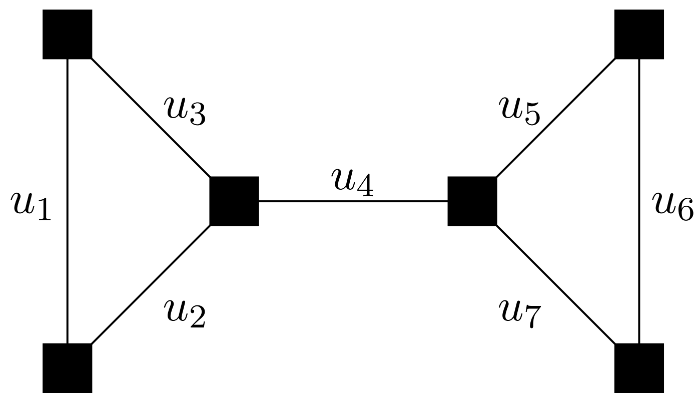
- Let C be a cycle code defined by a parity-check matrix $\mathbf{H}$ having normal graph $N \triangleq N(\mathbf{H})$.
- Let $n = n(N)$ be the number of edges of N.
- Let $\zeta_N(u_1, \ldots, u_n)$ be the edge zeta function of N.
- Then

  the monomial $u_1^{p_1} \ldots u_n^{p_n}$ has a nonzero coefficient
  in the Taylor series expansion of ζ_N

  if and only if

  the corresponding exponent vector $(p_1, \ldots, p_n)$
  is an unscaled pseudo-codeword for C.

Relationship Pseudo-Codewords and Edge Zeta Function (Part 2: Example)

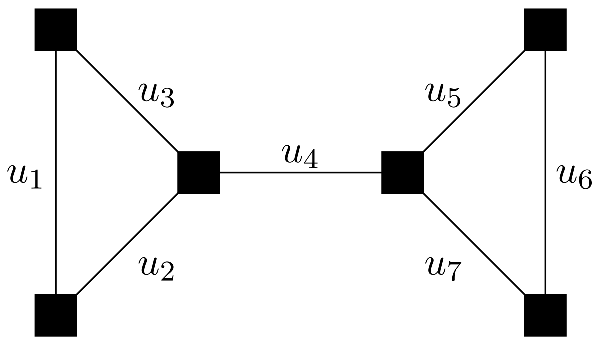


This normal graph N has the following inverse edge zeta function:

$$\zeta_N(u_1, \ldots, u_7) = \frac{1}{\det(\mathbf{I}_{14} - \mathbf{UM})}$$

$$= \frac{1}{\begin{array}{l} 1 - 2u_1u_2u_3 + u_1^2u_2^2u_3^2 - 2u_5u_6u_7 + 4u_1u_2u_3u_5u_6u_7 - 2u_1^2u_2^2u_3^2u_5u_6u_7 \\ -4u_1u_2u_3u_4^2u_5u_6u_7 + 4u_1^2u_2^2u_3^2u_4^2u_5u_6u_7 + u_5^2u_6^2u_7^2 - 2u_1u_2u_3u_5^2u_6^2u_7^2 \\ +u_1^2u_2^2u_3^2u_5^2u_6^2u_7^2 + 4u_1u_2u_3u_4^2u_5^2u_6^2u_7^2 - 4u_1^2u_2^2u_3^2u_4^2u_5^2u_6^2u_7^2 \end{array}}$$

Relationship Pseudo-Codewords and Edge Zeta Function (Part 3: Example)



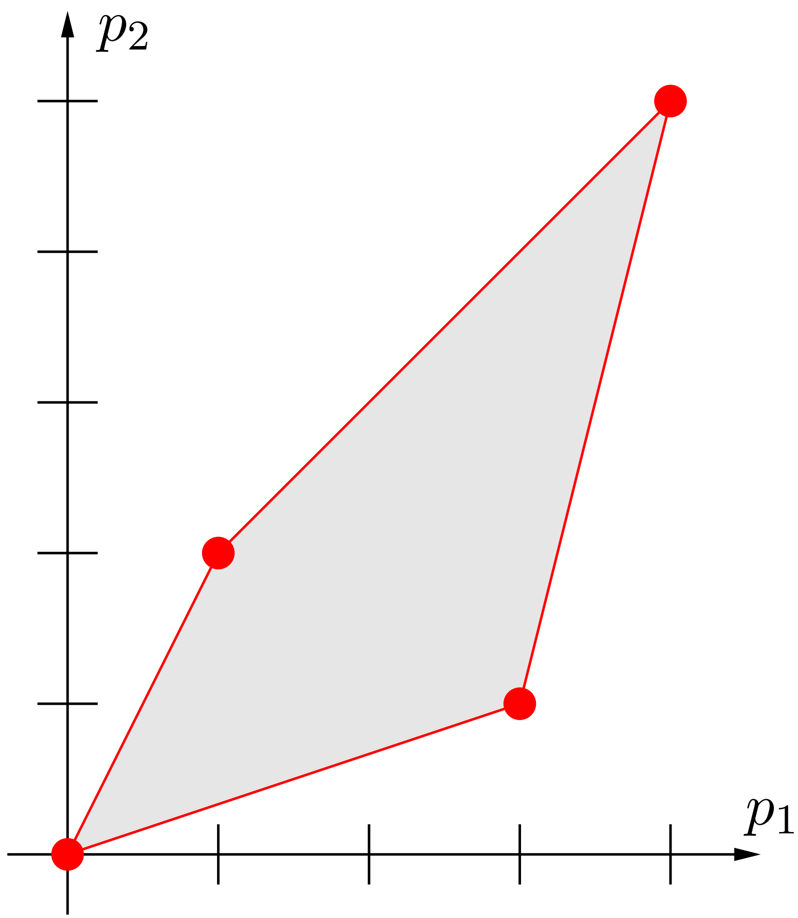
The Taylor series exansion is

$$\zeta_N(u_1, \ldots, u_7)$$
$$= 1 + 2u_1u_2u_3 + 3u_1^2u_2^2u_3^2 + 2u_5u_6u_7$$
$$+ 4u_1u_2u_3u_5u_6u_7 + 6u_1^2u_2^2u_3^2u_5u_6u_7$$
$$+ 4u_1u_2u_3u_4^2u_5u_6u_7 + 12u_1^2u_2^2u_3^2u_4^2u_5u_6u_7$$
$$+ \cdots$$

We get the following exponent vectors:

$(0,0,0,0,0,0,0)$	codeword
$(1,1,1,0,0,0,0)$	codeword
$(2,2,2,0,0,0,0)$	pseudo-codeword (in $\mathbb{Z}$-span)
$(0,0,0,0,1,1,1)$	codeword
$(1,1,1,0,1,1,1)$	codeword
$(2,2,2,0,1,1,1)$	pseudo-codeword (in $\mathbb{Z}$-span)
$(1,1,1,2,1,1,1)$	pseudo-codeword (not in $\mathbb{Z}$-span)
$(2,2,2,2,1,1,1)$	pseudo-codeword (in $\mathbb{Z}$-span)

The Newton Polytope of a Polynomial



Here: $P(u_1, u_2)$

$= u_1^0 u_2^0 + 3u_1^1 u_2^2 + 4u_1^3 u_2^1 - 2u_1^4 u_2^5$

Definition:

The Newton polytope of a polynomial $P(u_1, \ldots, u_n)$ in n indeterminates is the convex hull of the points in n-dimensional space given by the exponent vectors of the nonzero monomials appearing in $P(u_1, \ldots, u_n)$.

Similarly, we can associate a polyhedron to a power series.

Characterizing the Fundamental Cone Through the Zeta Function

Collecting the results from the previous slides we get:

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Proposition: Let $\mathcal{C}$ be some cycle code with parity-check matrix $\mathbf{H}$ and normal factor graph $N(\mathbf{H})$.

The Newton polyhedron of the edge zeta function of $N(\mathbf{H})$

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However, what is the meaning of the coefficients of the monomials in the Taylor series expansion of the edge zeta function?

Connecting the Bethe Entropy and the Edge Zeta Function of a Cycle Code

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Theorem: Let $\boldsymbol{\omega}$ be a pseudo-codeword with rational components. Then

$$\frac{\mathrm{d}}{\mathrm{d}t} H_{\mathrm{Bethe}}(t\boldsymbol{\omega})\bigg|_{t \downarrow 0} \quad = \quad G_{\mathrm{coeff}}\big(\zeta_{\mathsf{N}(\mathbf{H})}, \boldsymbol{\omega}\big).$$

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- The first term is the directional derivative of the induced Bethe entropy at the origin in the direction of $\boldsymbol{\omega}$.
- The second term is the growth rate of the coefficients of the monomials that appear in the Taylor series expansion of the edge zeta function $\zeta_{\mathsf{N}(\mathbf{H})}$ and whose exponent vector equals a positive multiple of $\boldsymbol{\omega}$.

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to be the asymptotic growth rate of the coefficients of the monomials that appear in the Taylor series expansion of the edge zeta function ζ_{G} of the graph G and whose exponent vector equals a positive multiple of $\boldsymbol{\omega}$.

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to be the asymptotic growth rate of the coefficients of the monomials that appear in the Taylor series expansion of the edge zeta function ζ_{G} of the graph G and whose exponent vector equals a positive multiple of $\boldsymbol{\omega}$.

For example, if $\boldsymbol{\omega} = \frac{1}{2} \cdot (1, 1, 1, 2, 1, 1, 1)$ then we consider the asymptotic growth rate of the coefficients of the monomials

$u_1^1 u_2^1 u_3^1 u_4^2 u_5^1 u_6^1 u_7^1, \;\; u_1^2 u_2^2 u_3^2 u_4^4 u_5^2 u_6^2 u_7^2, \;\; u_1^3 u_2^3 u_3^3 u_4^6 u_5^3 u_6^3 u_7^3, \;\; \ldots$

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- The first term is the directional derivative of the induced Bethe entropy at the origin in the direction of ω.
- The second term is a scaled version of the entropy rate of some time-invariant Markov process that is associated with ω.
- The third term is the growth rate of the coefficients of the monomials that appear in the Taylor series expansion of the edge zeta function and whose exponent vector equals a positive multiple of ω.

Another Result about the Bethe Entropy around the Origin

Theorem (second-order derivative result of the Bethe entropy):
The larger the eigenvalue gap between the largest and second-largest eigenvalue of the adjacency matrix of the normal factor graph, the larger the curvature of the Bethe entropy around the origin.

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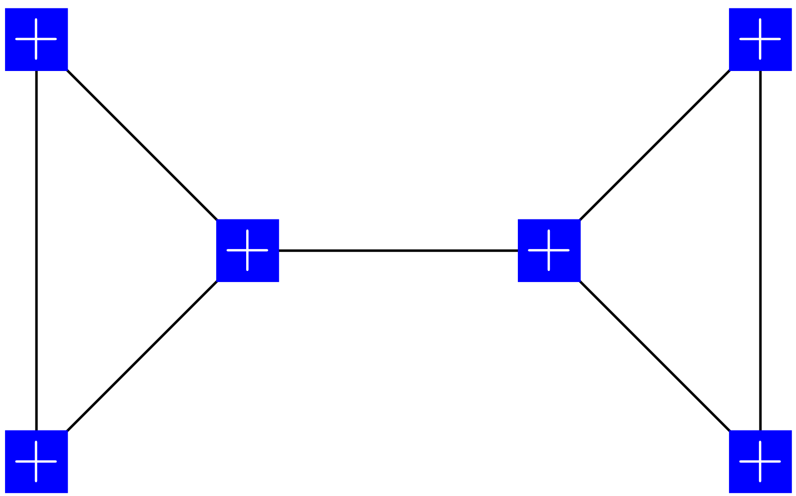
$\Rightarrow$ Use so-called Ramanujan graphs to obtain graphical models whose Bethe entropy has maximal curvature around the origin.

Connection to Results by Watanabe and Fukumizu

It turns out that the our setup and the setup by [Watanabe/Fukumizu, this workshop / NIPS 2009] are dual to each other in the sense that the sets of valid configurations are given by dual Forney-style factor graphs.



our setup

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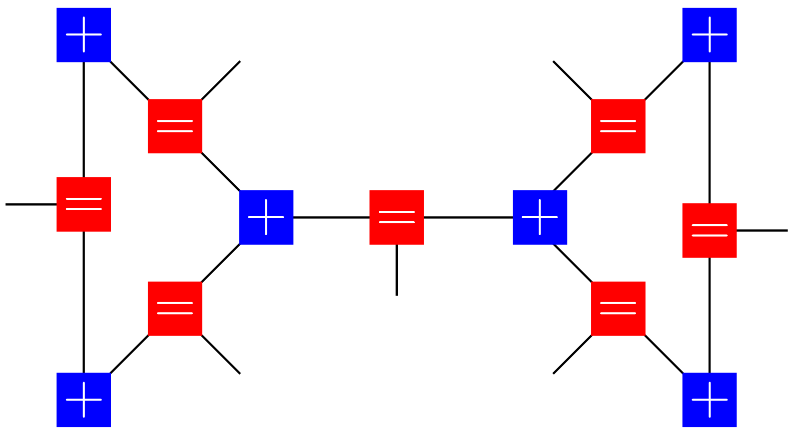

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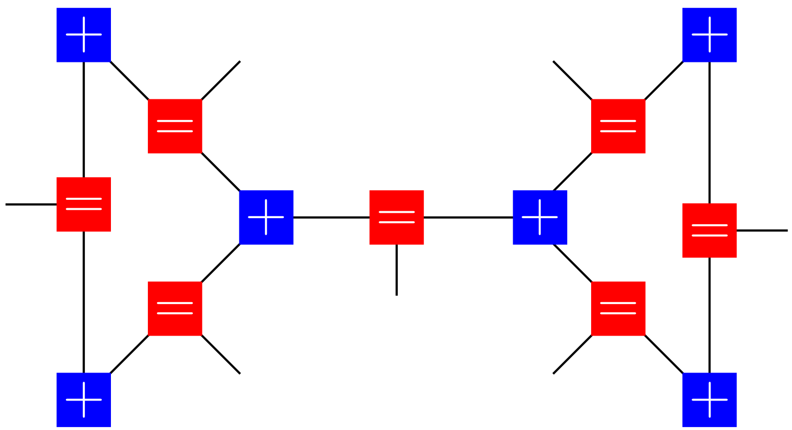

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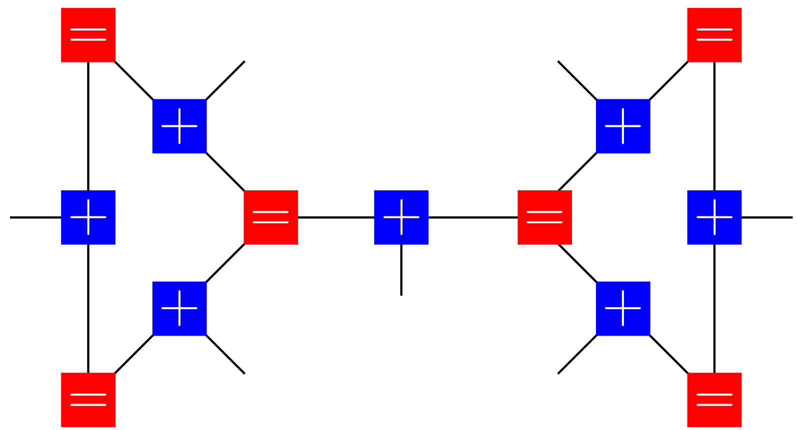

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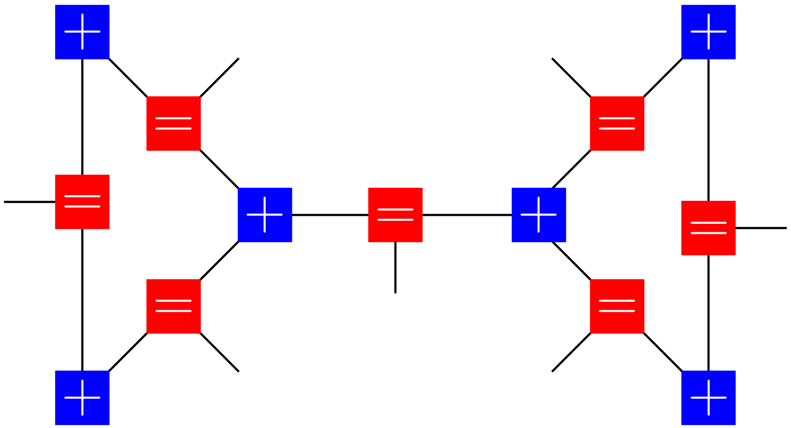

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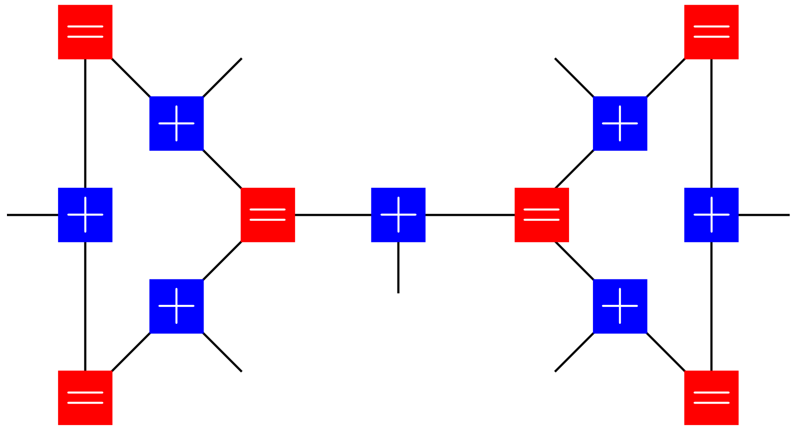

Watanabe/Fukumizu setup

However, the type of obtained results are quite different.

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  $\Rightarrow$ See also the talk on Friday morning by Watanabe.


Thank you!
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